A LIGHT BULB THEOREM FOR MULTI-DISKS

MAXIMILIAN HANS

Abstract. Given a 4-dimensional manifold M, together with an n-component link $\ell : \mathbb{S}^{1,n} \hookrightarrow \partial M$, for which each component is equipped with an embedded geometric dual sphere, we give a classification of isotopy classes of slice multi-disks by studying the homotopy type of the embedding space $\mathrm{Emb}_\ell(\mathbb{D}^{2,n}, M)$.

Contents

Date: 20th July 2024.

I. INTRODUCTION

The higher dimensional analogue to classical knot theory is the study of *knotted* surfaces in 4-manifolds. Using Casson handles (later skyscrapers in the work with Quinn) and the presence of algebraic dual spheres, Freedman constructed topological embeddings of surfaces into 4-manifolds and determined their topological isotopy classes in some cases. His construction is entirely topological, and for many years the question about analogous results in the smooth category was completely open. Gabai's Light Bulb Theorem in dimension 4, see [Gab20], was a major result into the insight of isotopy classes of surfaces in 4-manifolds. Two homotopic 2-spheres in a 4-manifold such that $\pi_1(M)$ has no elements of order 2, that share a common geometric dual sphere, are already isotopic. It has been known for a while that, without a common dual sphere, such results fail. In [ST22], Schneiderman and Teichner extended Gabai's Light Bulb Theorem to arbitrary fundamental groups. As a corollary, they showed that two homotopic 2-spheres in a 4-manifold, sharing a common geometric dual sphere, are isotopic if and only if their Freedman-Quinn invariant vanishes.

I.1. STRUCTURE AND STATEMENT OF THE RESULTS

We present the structure and main results, encouraging the reader to come back to this section every now and then, to take a break from the technicalities and maintain an overview of the content. This work is heavily inspired, motivated by and built upon the work of Kosanović and Teichner in [KT23b] and [KT23a], and is extending some of the results to the setting of multi-disks.

Let M be a d-dimensional, compact, oriented, connected manifold with non-empty boundary ∂M. Consider an n-component link $\ell : \mathbb{S}^{k-1,n} \hookrightarrow \partial M$ such that every component $\ell_i : \mathbb{S}^{k-1} \hookrightarrow \partial M$ has its own framed geometric dual $G_i : \mathbb{S}^{d-k} \hookrightarrow \partial M$. This means that $\ell_i \pitchfork G_i = \mathrm{pt}$ and G_i has trivialised normal bundle. Furthermore, we require that $\ell_i \cap G_j = \emptyset$ and $G_i \cap G_j = \emptyset$ for $i \neq j$. Considering all components together, we say that the link $G : \mathbb{S}^{d-k,n} \hookrightarrow \partial M$ is a framed geometric dual link to ℓ and we are in the setting with a dual. Note that this setting is rather restrictive. In fact, the existence of such a link ℓ together with a framed geometric dual link G implies that $\partial M \cong \#_{i=1}^{n}(\mathbb{S}^{k-1} \times \mathbb{S}^{d-k})\# W$ for some closed, oriented $(d-1)$-manifold W. This can be seen via the attaching map of the $(d-1)$-cell on the *transverse* wedge $\mathbb{S}^{k-1} \vee_{\pitchfork} \mathbb{S}^{d-k}$ in the boundary of M. Here, the fact that each dual sphere is framed is crucial. Notice that W need not be connected and hence the information on which path-component to form the connected sum on is important, but usually suppressed in favour of less cluttered notation.

We will later focus on the case of a link $\ell : \mathbb{S}^{1,n} \hookrightarrow \partial M$ where M is of dimension 4. In this case, the geometric dual link $G : \mathbb{S}^{2,n} \hookrightarrow \partial M$ automatically has trivial normal bundle since every real line bundle, classified by $H^1(-;\mathbb{Z}/2)$, over the sphere $\mathbb{S}^2$ is trivial. In this case, a Heegaard diagram of ∂M yields another point of view on why ∂M must contain connected sums of $\mathbb{S}^1 \times \mathbb{S}^2$.

The object of interest is the set of isotopy classes of multi-disks $\mathbb{D}^{k,n} \hookrightarrow M$ that agree with the n-component link ℓ on the boundary. Our approach is to study the homotopy type of the entire embedding space $\mathrm{Emb}_\ell(\mathbb{D}^{k,n}, M)$. Elements in this space are precisely embeddings $\mathbb{D}^{k,n} \hookrightarrow M$ that agree with ℓ on the boundary.

In Section II, we begin with a short discussion on manifolds with corners, stating well-known results which are serving as key inputs. After introducing important embedding spaces, we give a proof of the following theorem which is an essential ingredient in the theory.

Theorem A. *Let M be a d-dimensional, compact, oriented manifold with non-empty boundary ∂M and $\ell : \mathbb{S}^{k-1,n} \hookrightarrow \partial M$ an n-component link with a framed dual link $G : \mathbb{S}^{d-k,n} \hookrightarrow \partial M$. Any choice of a basepoint $\mathrm{U} \in \mathrm{Emb}_{\ell^\varepsilon}(\mathbb{D}^{k,n}, M)$ leads to an inverse pair of homotopy equivalences*

$$\mathrm{Emb}_{\ell^\varepsilon}(\mathbb{D}^{k,n}, M) \underset{\mathfrak{a}_{\mathrm{U}}}{\overset{\mathfrak{f}^\varepsilon_{\mathrm{U}}}{\rightleftarrows}} \Omega\,\mathrm{Emb}^\varepsilon_{u_0}(\mathbb{D}^{k-1,n}, M_G).$$

The embedding spaces in question are defined in Section II. In particular, the embedding spaces $\mathrm{Emb}_\ell(\mathbb{D}^{k,n}, M)$ and $\mathrm{Emb}_{\ell^\varepsilon}(\mathbb{D}^{k,n}, M)$ are weakly equivalent. Note that the existence of the basepoint U implies that the embedding space $\mathrm{Emb}_\ell(\mathbb{D}^{k,n}, M)$ is non-empty to begin with. Of course, it is an interesting question itself to ask when this embedding space is non-empty, which is the question about *sliceness* of the link ℓ in the d-manifold M. In the setting with a dual link and the dimensional restriction $d \geq 2k$, this is exactly the case if and only if ℓ_i is null-homotopic in M, see Proposition 12 for a discussion. The notable point of the above theorem is that, in the presence of dual spheres, the homotopy type of the embedding space of interest can be understood by studying the homotopy type of another embedding space with increased codimension.

We then turn our attention towards the special case of links $\ell : \mathbb{S}^{1,n} \hookrightarrow \partial M$ in the boundary of a 4-manifold M. By the previous reduction, we study the homotopy type of the embedding space $\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$ of arcs in the 4-manifold M. In Section III, we introduce and alter the work by Dax on comparing the homotopy type of embedding spaces to that of immersion spaces. This leads to a *geometric Dax isomorphism* Dax, together with an explicit inverse $\mathfrak{t}$.

$$\pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \underset{\mathfrak{t}}{\overset{\mathsf{Dax}}{\rightleftarrows}} \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)]$$

We further discuss the homotopy type of immersion spaces via Hirsch-Smale theory in IV. This leads to the following theorem.

Theorem B. *Let M be 4-dimensional, compact, oriented, connected manifold with non-empty boundary ∂M. For a whiskered embedding u as a basepoint, there is a central group extension*

$$\begin{tikzcd}
\mathbb{Z}[\pi_1(M)^{\dagger,n}]/\mathrm{im}(\mathsf{dax}_u) \arrow[r, tail, shift left, "\partial\mathfrak{r}"] & \pi_1(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \arrow[l, dashed, shift left, "\mathsf{Dax}"] \arrow[r, two heads] & \prod_{i=1}^{n} \pi_2(M).
\end{tikzcd}$$

This identifies the subgroup $\pi_1^D(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \leqslant \pi_1(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$ of loops of embeddings that are null-homotopic in the mapping space $\mathrm{Map}_\partial(\mathbb{D}^{1,n}, M)$ with the abelian group $\mathbb{Z}[\pi_1(M)^{\dagger,n}]/\mathrm{im}(\mathsf{dax}_u)$.

A phenomenon this sequence detects is the following. Given any such 4-manifold M, in the case of $n = 1$, every homotopy class of loops of neat embeddings $\mathbb{D}^1 \hookrightarrow M$ contains exactly one isotopy class. Already in the case of $n = 2$, each such homotopy class contains countably many different isotopy classes. These can be realised by the restricted realisation map $\partial\mathfrak{r}$ as introduced in Section III.

After a discussion on forgetting the ε-augmentation in the beginning of Section V, we then tie up loose ends to obtain the following theorem which can be thought of as the main result regarding the classification of isotopy classes of multi-disks in 4-manifolds in the presence of a geometric dual link.

Theorem C. *Let M be 4-dimensional, compact, oriented, connected manifold with non-empty boundary ∂M. In the setting with a geometric dual link, there is a short exact sequence of sets*

$$\begin{tikzcd}
\mathbb{Z}[\pi_1(M_G)^{\dagger,n}]/\mathsf{dax}_u(M_G) \arrow[r, tail, shift left, "\partial\mathfrak{r}"] & \pi_0(\mathrm{Emb}_\ell(\mathbb{D}^{2,n}, M), \mathrm{U}) \arrow[l, dashed, shift left, "\mathsf{Dax}"] \arrow[r, two heads] & \mathbb{Z}^n \times \prod_{i=1}^{n} \pi_2(M_G).
\end{tikzcd}$$

This result can be thought of as a combination of Theorem A and Theorem B.

I.2. NOTATION AND CONVENTIONS

If not stated otherwise, we always assume manifolds to come equipped with a smooth structure, and maps between manifolds are assumed to be smooth. Furthermore, loop spaces ΩX are based. Given two smooth manifolds M and N, $\mathrm{Emb}(M, N)$ denotes the space of smooth embeddings viewed as a subspace of the smooth mapping space $C^\infty(M, N)$ equipped with the Whitney C^∞-topology. Similarly, the immersion space $\mathrm{Imm}(M, N)$ comes equipped with the Whitney C^∞-topology. The original definition of the Whitney C^∞-topology comes in the form of convergence of partial derivatives on compact sets. This is most effectively described as a subspace of the space of sections of jet bundles. The standard reference is [GG73, Chapter 2, Section 3], [Hir76, Chapter 2], or [Mic80, Chapter 1, Chapter 4]. Immersions are assumed to be *generic*, that is, a smooth map with only finitely many points of transverse intersection.

- $\mathbb{S}^k$ denotes the k-dimensional sphere, $\mathbb{S}^{k,n} := \coprod_{i=1}^{n} \mathbb{S}^k$ the n-component disjoint union of k-spheres, also called a multi-sphere.

- $\mathbb{D}^k$ denotes the k-dimensional disk, $\mathbb{D}^{k,n} := \coprod_{i=1}^{n} \mathbb{D}^k$ the n-component disjoint union of k-disks, also called a multi-disk.
- An n-component link ℓ of k-spheres in a smooth manifold X is a smooth embedding $\ell : \mathbb{S}^{k,n} \hookrightarrow X$.
- $\mathrm{Emb}_\partial(X, Y)$ denotes the space of neat embeddings from a smooth ℓ-manifold X with boundary into a smooth d-manifold Y with boundary. A smooth map $f : X \to Y$ is *neat* if it is transverse to ∂Y and $f^{-1}(\partial Y) = \partial X$. Sometimes the boundary condition might change, but embeddings with a boundary condition are generally understood to be neat, except in the case of multi-half-disks. For an explicit definition, see Definition 3.
- ᗡ^k denotes a k-dimensional half-disk as defined in Definition 8, $\text{ᗡ}^{k,n} := \coprod_{i=1}^{n} \text{ᗡ}^k$ the n-component disjoint union of k-dimensional half-disks, also called a multi-half-disk.
- $\mathbb{I}$ denotes the standard closed interval.

Concatenation (for example of paths) is denoted by $\cdot$ and reads from "left to right", whereas usual composition is denoted by $\circ$ and reads from "right to left".

Acknowledgements. I wholeheartedly thank Peter Teichner for his guidance and support, for the countless inspiring and motivating conversations, and for his encouragement to ask questions, regardless of their nature. His geometric approach to mathematics beautifully illuminates the details surrounding an idea, profoundly shaping my understanding and perspective on 4-dimensional manifolds. I am very fortunate to have had Peter Teichner as my advisor.

I thank Arunima Ray for her continuous encouragement, both inside and outside of mathematics. Throughout the years, she patiently taught me a great deal of what I know about low-dimensional topology. I am thankful for her giving me the opportunity to travel to India to attend a conference, and for all the life advice she gave me, especially during times I needed it the most.

I wish to thank Jan Sneeuw for all the great conversations about everything under the sun, usually over a pint of beer at a pub. Furthermore, I thank him for proofreading an earlier version of this document.

I am incredibly grateful for having Tesha in my life. Despite us rarely being in the same time zone, she brings me much joy, needed balance, warmth, and much more every single day.

Last but not least, I thank my family for supporting me throughout my whole life, always showing interest and involvement, and providing a safety net as I am navigating life.

II. ON EMBEDDINGS OF MULTI-DISKS AND MULTI-HALF-DISKS

II.1. EMBEDDINGS OF MANIFOLDS WITH CORNERS

In this section, we will review the necessary background on manifolds with corners. Restriction maps of embeddings are discussed, which will serve as a major tool for obtaining fibration sequences. For a careful introduction to manifolds with corners, we refer the reader to [Mic80, Chapter 2].

Definition 1. A *d-manifold with corners* consists of the following data.

- A topological space X which is Hausdorff and second countable.
- A collection $\{U_x, \varphi_x\}_{x\in X}$ with $U_x \subseteq X$ an open neighbourhood of x, and Diff-compatible charts $\varphi_x : U_x \longrightarrow \mathbb{R}^d_{(q)} := \mathbb{R}^q \times [0,\infty)^{d-q}$ for some $0 \leq q \leq d$, centred around x.

The charts being Diff-compatible means that every transition map defined on an open subset of $\mathbb{R}^d_{(q)}$ extends to a smooth map on an open subset of $\mathbb{R}^d$. This allows us to endow the set $C^\infty(Y, X)$ of smooth maps between manifolds with corners with the Whitney C^∞-topology. An *embedding* $\mathscr{E} : Y \hookrightarrow X$ of manifolds with corners is a smooth map such that $\mathscr{E}(Y)$ is a submanifold of X, and the induced corner structure makes the map $\mathscr{E}$ a diffeomorphism onto its image. An *immersion* of manifolds with corners is a smooth map that is locally an embedding. Spaces of embeddings $\mathrm{Emb}(Y, X)$ and immersions $\mathrm{Imm}(Y, X)$ are subspaces of $C^\infty(Y, X)$ and similarly carry the Whitney C^∞-topology.

The connected components of $X_{(q)} := \{x \in X : x \longleftrightarrow 0 \in \mathbb{R}^d_{(q)} \text{ in some chart}\}$ are called q-faces of X. Each q-face is a smooth submanifold of dimension q that has empty boundary. In Definition 1 above, the case $q = d$ corresponds to charts around points in the *interior* of the manifold X, while the case $q = d - 1$ corresponds to charts around points in the *smooth boundary* $\partial_s X$ of X. Furthermore, the closure of $X_{(d)}$ is the manifold X itself, the closure of $X_{(d-1)}$ is the whole boundary ∂X. Generally, we can decompose $\partial X = \partial_s X \cup \mathfrak{c}X$ where $\mathfrak{c}X$ denotes the *set of corners* of X.

Example 2. Let M and N be two regular manifolds with non-empty boundary. Then $M \times N$ is a manifold with corners. Namely, $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$ and $\mathfrak{c}(M \times N) = \partial M \times \partial N$.

Turning our attention back to embeddings, we will now carefully define embeddings with boundary conditions for manifolds with corners. Specifically, we say that two embeddings $\mathscr{E}_1 : Y \hookrightarrow X$ and $\mathscr{E}_2 : Y \hookrightarrow X$ of manifolds with corners have the same *incidence relations* if for each point $p \in Y$, and each face $F_X \subseteq X$, we have $\mathscr{E}_1(p) \in F_X$ if and only if $\mathscr{E}_2(p) \in F_X$. This means that there is a consistent choice of which faces of Y get mapped to which faces of X. For example, in the case of manifolds with boundary, and the set of corners being empty, this could mean that boundary gets mapped to boundary.

Definition 3. Let Y and X be manifolds with corners, and fix an embedding $y : Y \hookrightarrow X$. In particular, the embedding space $\mathrm{Emb}(Y, X)$ is non-empty. Consider the subspace

$$\mathrm{Emb}(Y, X; y) \subseteq \mathrm{Emb}(Y, X)$$

consisting of embeddings $\mathscr{E} : Y \hookrightarrow X$ such that $\mathscr{E}$ has the same incidence relations as y. For a closed subset $Y' \subseteq Y$, the subspace

$$\mathrm{Emb}_{Y'}(Y, X; y) \subseteq \mathrm{Emb}(Y, X; y)$$

is the space of such embeddings that agree with y on Y'. In that case, we call y the boundary condition along Y'.

With the definition at hand, usual manifolds with boundary are considered to be manifolds with corners. Let X, and $Y \supseteq Z \supseteq Z'$, be compact manifolds with corners, and consider embeddings

$$\begin{tikzcd} Z' \arrow[r, hook] & Z \arrow[r, hook, "i"] \arrow[rr, bend left, "z=y\circ i"] & Y \arrow[r, hook, "y"] & X \end{tikzcd}$$

with $i : Z \hookrightarrow Y$ being the chosen inclusion. Following the discussion given in [KT23b, Section 2.2.1], we call a subset $Y' \subseteq Y$ a *local normal tube* to $Z \subseteq Y$ along Z', if $Y' \cap Z = Z'$ and there exists a tubular neighbourhood $V \subseteq Y$ of Z in Y such that $Y' \cap V = \mathsf{proj}^{-1}(Z')$ with the canonical projection map $\mathsf{proj} : V \longrightarrow Z$.

Theorem 4 ([Cer61, p. 294, p. 298])**.** *In the setting as described above, the following restriction maps*

- $\mathsf{ev}_Z : \mathrm{Emb}(Y, X; y) \longrightarrow \mathrm{Emb}(Z, X; z)$
- $\mathsf{ev}_Z : \mathrm{Emb}_{Y'}(Y, X; y) \longrightarrow \mathrm{Emb}_{Z'}(Z, X; z)$

are locally trivial.

Locally trivial maps over paracompact Hausdorff spaces are Hurewicz fibrations, a result known as the Hurewicz-Huebsch theorem on local-to-global Hurewicz fibrations, see for example [Hue55]. Embedding spaces are paracompact as metrisable spaces and hence the locally trivial maps ev_Z in Cerf's theorem are, in fact, Hurewicz fibrations. See Remark 11 for further comments on the topological properties of embedding spaces. The above theorem can be seen as a family version of the classical ambient isotopy theorem which is obtained by the induced surjection on π_0. More concretely, the fact that it is a Hurewicz fibration yields a lift in the following diagram.

$$\begin{tikzcd} \mathrm{Emb}_{Y'}(Y, X; y) \arrow[r, "\mathsf{ev}_Z"] & \mathrm{Emb}_{Z'}(Z, X; z) \\ \{0\} \arrow[u] \arrow[r, hook] & \mathbb{I} \arrow[u] \arrow[ul, dashed] \end{tikzcd}$$

Proposition 5 ([Cer61, p. 331 , p. 337])**.** *If $Z' = Y' \cap Z$ is the closure of a codimension 1 face, then the inclusion $\mathrm{Emb}_{Z\cup Y'}(Y, X; y) \hookrightarrow \mathrm{Emb}_Z(Y, X; y)$ is a weak homotopy equivalence.*

Let us briefly discuss the ideas and results that go into proving Theorem 4. The key ingredient requires a discussion on G-equivariant maps. For a topological group G and a space X with a continuous G-action $G \curvearrowright X$, we say that X admits a local G-section at a point $x \in X$ if the orbit map sending a group element $g \in G$ to $g.x$ in the orbit of x has a local section at x. That is, there exists a neighbourhood $U \ni x$ and a map $s_U : U \longrightarrow G$ such that $s(u).x = u$ for all $u \in U$. The first key result is the following lemma, sometimes called the Cerf-Palais fibration criterion, which appeared independently in [Cer61, p. 240] and [Pal60, Theorem A].

Lemma 6 (Cerf-Palais fibration criterion)**.** *If $p\colon E \longrightarrow X$ is a G-equivariant map and X admits local G-sections at all points, then p is a locally trivial map.*

Sketch of the proof. One can simply choose a local trivialisation from $U \times p^{-1}(x)$ to $p^{-1}(U)$ by sending (u, v) to $s_U(u).v$. ☐

As we are considering embedding spaces, the natural action is given by post-composition with elements in the diffeomorphism group $\mathrm{Diff}(X)$ for the first case in Theorem 4. For the second case, one considers the group $\mathrm{Diff}_{z(Z')}(X)$ of diffeomorphisms that restrict to the identity on $z(Z')$. The evaluation maps ev_Z are evidently $\mathrm{Diff}(X)$- and $\mathrm{Diff}_{z(Z')}(X)$-equivariant. Focusing on the second case, the second key ingredient is given by the following theorem, which first appeared in [Cer61, p. 293].

Theorem 7 (Parametrised Isotopy Extension Theorem)**.** *The embedding space $\mathrm{Emb}_{Z'}(Z, X; z)$ admits local $\mathrm{Diff}_{z(Z')}(X)$-sections at all points.*

Theorem 4 then follows immediately from Lemma 6 and Theorem 7.

II.2. MULTI-HALF-DISKS AND IMPORTANT EMBEDDING SPACES

We will now focus on an important class of manifolds with corners, namely multi-half-disks. These spaces have faces of codimension 0, the interior, 1, the boundary, and 2, the corners. In the following, we set $k \geq 2$.

Definition 8. We call the space $ᗡ^k := \{x \in \mathbb{R}^k : \|x\| \leq 1, x_1 \leq 0\}$ a *half-disk of dimension k*, and the n-fold disjoint union $ᗡ^{k,n} := \coprod_{i=1}^n ᗡ^k$ a *multi-half-disk of dimension k*. We define the following important subspaces.

- $\mathbb{D}_-^{k-1} := \{x \in ᗡ^k : \|x\| = 1\}$ is the *outer boundary component* of the half-disk, and $\mathbb{D}_-^{k-1,n} := \coprod_{i=1}^n \mathbb{D}_-^{k-1} \subseteq ᗡ^{k,n}$ is the outer boundary component of the multi-half-disk.
- $\mathbb{D}_-^{\varepsilon,k-1} := \{x \in ᗡ^k : \|x\| \geq 1 - \varepsilon\}$ is the *outer boundary ε-neighbourhood* of the half-disk, and $\mathbb{D}_-^{\varepsilon,k-1,n} := \coprod_{i=1}^n \mathbb{D}_-^{\varepsilon,k-1} \subseteq ᗡ^{k,n}$ is the outer boundary ε-neighbourhood of the multi-half-disk.

- $\mathbb{D}_+^{k-1} := \{x \in \text{ᗡ}^k : x_1 = 0\}$ is the *inner boundary component* of the half-disk, and $\mathbb{D}_+^{k-1,n} := \coprod_{i=1}^n \mathbb{D}_+^{k-1} \subseteq \text{ᗡ}^{k,n}$ is the inner boundary component of the multi-half-disk.
- $\mathbb{D}_+^{\varepsilon,k-1} := \{x \in \text{ᗡ}^k : x_1 \geq -\varepsilon\}$ is the *inner boundary ε-neighbourhood* of the half-disk, and $\mathbb{D}_+^{\varepsilon,k-1,n} := \coprod_{i=1}^n \mathbb{D}_+^{\varepsilon,k-1} \subseteq \text{ᗡ}^{k,n}$ is the inner boundary ε-neighbourhood of the multi-half-disk.

Furthermore, we denote by $\mathbb{S}_0^{k-2} = \mathbb{D}_-^{k-1} \cap \mathbb{D}_+^{k-1}$ the unique corner of the half-disk, and the n-fold disjoint union $\mathbb{S}_0^{k-2,n} := \coprod_{i=1}^n \mathbb{S}_0^{k-2} \subseteq \text{ᗡ}^{k,n}$ are the n corners of the multi-half-disk. The union $\mathbb{D}_-^{\varepsilon,k-1,n} \cup \mathbb{D}_+^{\varepsilon,k-1,n} =: \partial^\varepsilon \text{ᗡ}^{k,n}$ is called the *prismatic collar*.

Note that the multi-half-disk $\text{ᗡ}^{k,n}$ is indeed a manifold with corners. There are exactly n k-faces, given by the interior $\dot{\text{ᗡ}}^{k,n}$, $2n$ $(k-1)$-faces, given by $\dot{\mathbb{D}}_-^{k-1,n}$ and $\dot{\mathbb{D}}_+^{k-1,n}$. Last but not least, there are n $(k-2)$-faces given by the corners of $\text{ᗡ}^{k,n}$, namely $\mathbb{S}_0^{k-2,n}$.

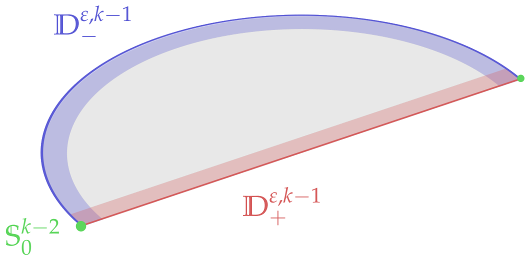


FIGURE 1. A half-disk ᗡ^k, separated into its faces, together with ε-neighbourhoods.

Important embedding spaces. We will now define various embedding spaces which are crucial for the upcoming section. Let V be a compact manifold with corners and $\mathrm{U} : V \hookrightarrow X$ a chosen fixed neat embedding, acting as the basepoint. Suppose V and X both have non-empty boundary. Then define the boundary condition $\ell := \mathrm{U}|_{\partial V} : \partial V \hookrightarrow \partial X$ and

$$\mathrm{Emb}_\ell(V, X) := \mathrm{Emb}_{\partial V}(V, X; \mathrm{U})$$

as the usual embedding space of neat embeddings $\mathscr{E} : V \hookrightarrow X$ agreeing with the basepoint U on the boundary. Thus for any such embedding $\mathscr{E}$, we have $\mathscr{E}|_{\partial V} = \ell$. Expanding the boundary condition ℓ on ∂V to a closed collar neighbourhood $\partial V \times [0, \varepsilon]$ yields the boundary condition $\ell^\varepsilon := \mathrm{U}|_{\partial V \times [0,\varepsilon]}$. The corresponding embedding space

$$\mathrm{Emb}_{\ell^\varepsilon}(V, X) := \mathrm{Emb}_{\partial V \times [0,\varepsilon]}(V, X; \mathrm{U})$$

consists of neat embeddings $\mathscr{E} : V \hookrightarrow X$ agreeing with U on the collar $\partial V \times [0, \varepsilon]$. Hence, $\mathscr{E}|_{\partial V \times [0,\varepsilon]} = \ell^\varepsilon$.

Lemma 9. *The inclusion* $\mathrm{Emb}_{\ell^\varepsilon}(V, X) \hookrightarrow \mathrm{Emb}_\ell(V, X)$ *is a weak homotopy equivalence.*

Proof. We apply Proposition 5, choosing $Z := \partial V =: Z'$ and $Y' := \partial V \times [0, \varepsilon]$. This immediately yields the result. □

Note that none of the used results depend on connectedness of V. Therefore, we can easily apply these results to our setting of embeddings of multi-disks into a manifold. The point is that the embedding space $\mathrm{Emb}_{\ell^\varepsilon}(V, X)$ allows us more control over how embeddings behave near the boundary. This will be useful in the proof of Theorem 10.

Generally, we are interested in the homotopy type of the embedding space $\mathrm{Emb}_{\ell}(\mathbb{D}^{k,n}, X)$ as discussed in Section I. The first key ingredient, Theorem A, is phrased in terms of the homotopy type of the embedding space $\mathrm{Emb}_{\ell^\varepsilon}(\mathbb{D}^{k,n}, X)$. Lemma 9 allows us to ignore this subtlety when calculating homotopy groups.

In the special case of multi-half-disks $\text{ᗡ}^{k,n}$, we allow one part of the boundary of the disk to lie in the interior of the target manifold X. Hence, we cannot assume embeddings to be neat in the classical sense but the boundary condition ℓ^ε still makes sense. The embedding space

$$\mathrm{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, X) := \mathrm{Emb}_{\partial^\varepsilon \text{ᗡ}^{k,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$$

is the space of embeddings that agree with a fixed basepoint $\mathrm{U} : \text{ᗡ}^{k,n} \hookrightarrow X$ on the prismatic collar $\partial^\varepsilon \text{ᗡ}^{k,n}$ as defined in Definition 8. Note that $\mathrm{U} : \text{ᗡ}^{k,n} \hookrightarrow X$ is understood to be neat on the outer boundary component of the multi-half-disk. In preparation of the proof of Theorem 10, we define the embedding space

$$\mathrm{Emb}_{u_0}(\mathbb{D}^{k-1,n}, X) := \mathrm{Emb}_{\mathbb{S}_0^{k-2,n}}(\mathbb{D}_+^{k-1,n}, X; u_+)$$

to be the embedding space of embeddings $\mathbb{D}^{k-1,n}$ into X that agrees with U on $\mathbb{S}_0^{k-2,n}$, the n corners of the multi-half-disk. The restriction $\mathrm{U}|_{\mathbb{D}_+^{k-1,n}}$ is denoted by u_+. By Lemma 9, this is weakly equivalent to the embedding space

$$\mathrm{Emb}_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X) := \mathrm{Emb}_{\mathbb{S}_0^{k-2,n} \times [0,\varepsilon]}(\mathbb{D}_+^{k-1,n}, X; u_+)$$

of embeddings that agree with U on $\mathbb{S}_0^{k-2,n} \times [0, \varepsilon]$. Last but not least, we require an embedding space of embeddings $\mathscr{E} : \mathbb{D}^{k-1,n} \hookrightarrow X$ that comes with the data of a push-off neighbourhood. Define the embedding space

$$\mathrm{Emb}^\varepsilon_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X) := \mathrm{Emb}_{\mathbb{D}_-^{\varepsilon,k-1,n} \cap \mathbb{D}_+^{\varepsilon,k-1,n}}(\mathbb{D}_+^{\varepsilon,k-1,n}, X; u_+^\varepsilon)$$

to be the space of embeddings $\mathscr{E} : \mathbb{D}^{k-1,n} \times [0, \varepsilon] \hookrightarrow X$ that agree with U on $\mathbb{D}_-^{\varepsilon,k-1,n} \cap \mathbb{D}_+^{\varepsilon,k-1,n}$. The restriction $\mathrm{U}|_{\mathbb{D}_+^{\varepsilon,k-1,n}}$ is denoted by u_+^ε. We call elements in $\mathrm{Emb}^\varepsilon_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X)$ "ε-augmented" multi-disks of dimension $(k-1)$.

II.3. COMPARING EMBEDDINGS OF MULTI-DISKS TO MULTI-HALF-DISKS

The goal of this section is to prove Theorem A. The proof is divided into two main parts and is completely analogous to the proof provided by Kosanović and Teichner in [KT23b, Theorem 3.2] for the case $n = 1$. The first ingredient is the following theorem. Note that this easily can be furthermore generalised to the setting of embedding of disks of varying dimension k, as

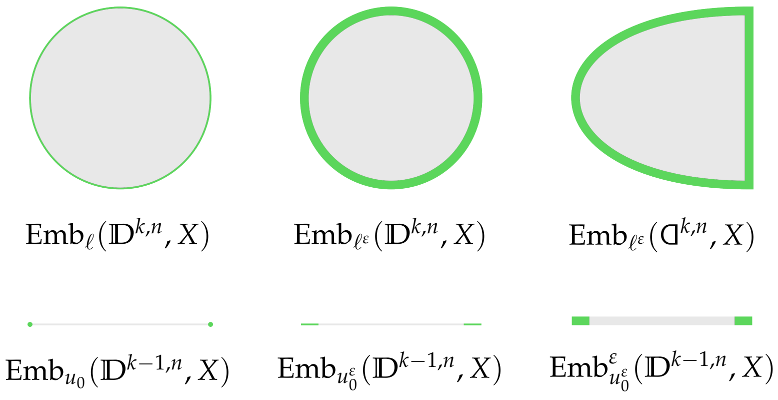


FIGURE 2. The boundary conditions of the various embedding spaces for the case $k = 2$ and $n = 1$.

long as every bounding $k-1$-sphere as a geometric dual sphere of dimension $d-k$. Since we do not need this generalisation, we omit it for the sake of readability.

Theorem 10. *For $k, d, n \geq 1$, X a smooth d-manifold with non-empty boundary ∂X and a chosen basepoint* $\mathrm{U} : \text{ᗡ}^{k,n} \hookrightarrow X$ *of* $\mathrm{Emb}_{s^{\varepsilon}}(\text{ᗡ}^{k,n}, X)$, *there are inverse homotopy equivalences*

$$\mathrm{Emb}_{\ell^{\varepsilon}}(\text{ᗡ}^{k,n}, X) \underset{\mathfrak{a}_{\mathrm{U}}}{\overset{\mathfrak{f}^{\varepsilon}_{\mathrm{U}}}{\rightleftarrows}} \Omega\,\mathrm{Emb}^{\varepsilon}_{u_0^{\varepsilon}}(\mathbb{D}^{k-1,n}, X).$$

Proof. By Theorem 4, there is a fibration sequence

$$\mathrm{ev}^{-1}_{\mathbb{D}^{\varepsilon,k-1,n}_{+}}(u^{\varepsilon}_{+}) \longrightarrow \mathrm{Emb}_{\mathbb{D}^{\varepsilon,k-1,n}_{-}}(\text{ᗡ}^{k,n}, X; \mathrm{U}) \xrightarrow{\mathrm{ev}_{\mathbb{D}^{\varepsilon,k-1,n}_{+}}} \mathrm{Emb}^{\varepsilon}_{u_0^{\varepsilon}}(\mathbb{D}^{k-1,n}, X)$$

by setting $Y' := \mathbb{D}^{\varepsilon,k-1,n}_{-} \subseteq \text{ᗡ}^{k,n} =: Y$ and $Z' := \mathbb{D}^{\varepsilon,k-1,n}_{-} \cap \mathbb{D}^{\varepsilon,k-1,n}_{+} \subseteq \mathbb{D}^{\varepsilon,k-1,n}_{+} =: Z$. This is sometimes called Cerf's half-disk trick. We will now identify the fibre space, after which we show that the total space is contractible. This will lead to a weak homotopy equivalence which we can upgrade to obtain an honest homotopy equivalence, proving the claim. We consider an element $\mathscr{E} \in \mathrm{Emb}_{\mathbb{D}^{\varepsilon,k-1,n}_{-}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$ which is an embedding $\mathscr{E} : \text{ᗡ}^{k,n} \hookrightarrow X$ that agrees with the embedding $\mathrm{U} : \text{ᗡ}^{k,n} \hookrightarrow X$ on an ε-neighbourhood $\mathbb{D}^{\varepsilon,k-1,n}_{-}$ of the outer boundary $\mathbb{D}^{k-1,n}_{-}$. The evaluation map $\mathrm{ev}_{\mathbb{D}^{\varepsilon,k-1,n}_{+}}$ restricts this embedding to an ε-neighbourhood $\mathbb{D}^{\varepsilon,k-1,n}_{+}$ of the inner boundary $\mathbb{D}^{k-1,n}_{+}$. This now agrees with the embedding U on the intersection $\mathbb{D}^{\varepsilon,k-1,n}_{-} \cap \mathbb{D}^{\varepsilon,k-1,n}_{+}$. This is illustrated in Figure 3 below, for the case $n = 1$. This reduction is sufficient as everything happens disjointedly.

Hence, considering the pre-image $\mathrm{ev}^{-1}_{\mathbb{D}^{\varepsilon,k-1,n}_{+}}(u^{\varepsilon}_{+})$ gives all embeddings that agree with U on $\mathbb{D}^{\varepsilon,k-1,n}_{-} \cup \mathbb{D}^{\varepsilon,k-1,n}_{+} = \partial^{\varepsilon}\text{ᗡ}^{k,n}$. This is precisely the space $\mathrm{Emb}_{\ell^{\varepsilon}}(\text{ᗡ}^{k,n}, X)$, giving the fibration

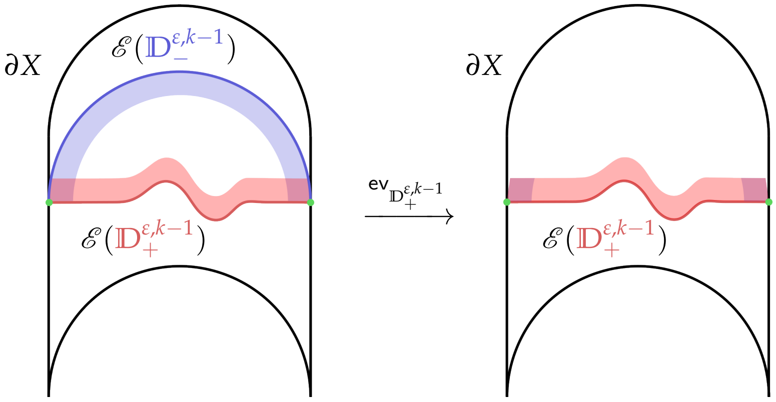


FIGURE 3. The action of the evaluation map on an embedding $\mathscr{E} : \text{ᗡ}^k \hookrightarrow X$ which agrees with U on an ε-neighbourhood $\mathbb{D}_-^{\varepsilon,k-1}$ of the outer boundary.

sequence

$$\mathrm{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, X) \longrightarrow \mathrm{Emb}_{\mathbb{D}_-^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U}) \xrightarrow{\mathrm{ev}_{\mathbb{D}_+^{\varepsilon,k-1,n}}} \mathrm{Emb}^{\varepsilon}_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X).$$

The claim is now that the total space $\mathrm{Emb}_{\mathbb{D}_-^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$ is contractible. Morally, the idea is to push a multi-half-disk radially into the neighbourhood on which it agrees with U. Then push it out radially to obtain U. The first part can be done in the following way. Namely, let $\varphi : [\varepsilon, 1] \longrightarrow \mathrm{Emb}_{\mathbb{D}_-^{\varepsilon/2,k-1,n}}(\text{ᗡ}^{k,n}, \text{ᗡ}^{k,n}; \mathrm{Id})$ be a path such that

- $\varphi_1 = \mathrm{Id}$,
- $\varphi_\varepsilon(\text{ᗡ}^{k,n}) \subseteq \mathbb{D}_-^{\varepsilon,k-1,n}$,
- $\varphi_t(\mathbb{D}_-^{\varepsilon,k-1,n}) \subseteq \mathbb{D}_-^{\varepsilon,k-1,n}$.

Note that the boundary condition on the embedding space means that we are fixing the multi-half-disk on a smaller neighbourhood of the outer boundary component. As discussed before, one should think about φ radially pushing the multi-half-disk into the $\varepsilon/2$-neighbourhood of the outer boundary. As a path of embeddings, this defines an isotopy. Equipped with the newly acquired isotopy, we obtain a homotopy

$$\mathrm{Emb}_{\mathbb{D}_-^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U}) \times [\varepsilon, 1] \longrightarrow \mathrm{Emb}_{\mathbb{D}_-^{\varepsilon/2,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$$

given by pre-composition $(\mathscr{E}, t) \mapsto \mathscr{E} \circ \varphi_t$. For each element $\mathscr{E} \in \mathrm{Emb}_{\mathbb{D}_-^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$, this amounts to a path from $\mathscr{E}$ itself to $\mathscr{E} \circ \varphi_\varepsilon = \mathrm{U} \circ \varphi_\varepsilon$, the latter given by the boundary condition. Currently, we run into a small problem. The target embedding space is larger than the one we want to define a retraction on. To circumvent this, we use the ambient isotopy theorem to extend the just defined isotopy $\mathrm{U} \circ \varphi_t$ to an ambient isotopy $\Phi : [\varepsilon, 1] \longrightarrow \mathrm{Emb}_{\nu\partial}(X, X)$ supported in a neighbourhood of the boundary of X. For $t \in [\varepsilon, 1]$, we define the first part of the retraction

$$\mathscr{R} : \mathrm{Emb}_{\mathbb{D}_{-}^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U}) \times \mathbb{I} \longrightarrow \mathrm{Emb}_{\mathbb{D}_{-}^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$$

on elements as $\mathscr{R}_t(\mathscr{E}) := \Phi_t \circ \mathscr{E} \circ \varphi_t$. For each multi-half-disk $\mathscr{E} : \text{ᗡ}^{k,n} \hookrightarrow X$, this is a path in $\mathrm{Emb}_{\mathbb{D}_{-}^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$, starting at $\mathscr{R}_\varepsilon(\mathscr{E}) = \Phi_\varepsilon \circ \mathscr{E} \circ \varphi_\varepsilon = \Phi_\varepsilon \circ \mathrm{U} \circ \varphi_\varepsilon$, as it agrees with U in an ε-neighbourhood of the outer boundary, to $\mathscr{R}_1(\mathscr{E}) = \mathrm{Id}_X \circ \mathscr{E} \circ \mathrm{Id}_{\text{ᗡ}^{k,n}}$. Since we have the equality $\mathscr{R}_\varepsilon(\mathscr{E}) = \mathscr{R}_\varepsilon(\mathrm{U})$ for any element $\mathscr{E} \in \mathrm{Emb}_{\mathbb{D}_{-}^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$, for the remaining $t \in [0, \varepsilon]$, we want to glue another path from $\mathscr{R}_\varepsilon(\mathrm{U}) = \Phi_\varepsilon \circ \mathrm{U} \circ \varphi_\varepsilon$ to U. This can be done by using a scaled version of the homotopy $\mathscr{R}$ but backwards and applied to U. For $t \in [0, \varepsilon]$, we define $\mathscr{R}_t(\varepsilon) := \mathscr{R}_{1+t-t/\varepsilon}(\mathrm{U})$. This yields the desired retraction and the space $\mathrm{Emb}_{\mathbb{D}_{-}^{\varepsilon,k-1,n}}(\text{ᗡ}^{k,n}, X; \mathrm{U})$ is contractible. Therefore, we obtain a weak homotopy equivalence

$$\mathrm{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, X) \simeq_w \Omega\, \mathrm{Emb}^{\varepsilon}_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X).$$

Since it is the total space that is contractible, the connecting-map in the long exact sequence of homotopy groups provides the weak homotopy equivalence. In that case, the weak homotopy equivalence can be upgraded to a pair of inverse homotopy equivalences. This process is discussed in [KT23b, Appendix A] and is a general fact for Hurewicz fibrations whose connecting map on homotopy groups is a weak equivalence. In this case, the connecting map can be identified using the ambient isotopy theorem. We simply give a description of the two homotopy inverse maps

$$\mathrm{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, X) \underset{\mathfrak{a}_{\mathrm{U}}}{\overset{\mathfrak{f}^{\varepsilon}_{\mathrm{U}}}{\rightleftarrows}} \Omega\, \mathrm{Emb}^{\varepsilon}_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X).$$

The map $\mathfrak{a}_{\mathrm{U}} : \Omega\, \mathrm{Emb}^{\varepsilon}_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X) \longrightarrow \mathrm{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, X)$ sends a loop of multi-disks $\mathbb{D}^{k-1,n}$ in X to an ambient isotopy which ends with a diffeomorphism taking the basepoint U to another multi-half-disk $\text{ᗡ}^{k,n}$ in X that agrees with U on its boundary. The map $\mathfrak{f}^{\varepsilon}_{\mathrm{U}} : \mathrm{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, X) \longrightarrow \Omega\, \mathrm{Emb}^{\varepsilon}_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X)$ is given by foliating a given multi-half-disk $\text{ᗡ}^{k,n}$ in X by a family of multi-disks $\mathbb{D}^{k-1,n}$, starting at $u_+ := \mathrm{U}|_{\mathbb{D}_{+}^{k-1,n}}$ until one arrives at $u_- := \mathrm{U}|_{\mathbb{D}_{-}^{k-1,n}}$, and then going back via a foliation of U to obtain a loop of multi-disks. $\square$

Remark 11. Without identifying the explicit homotopy inverses, another reason we can upgrade the weak equivalence to a homotopy equivalence is due to the fact that the spaces in question have the homotopy type of a CW-complex. Indeed, in [Pal66], Palais demonstrated that embedding spaces are *dominated* by CW-complexes, as embedding spaces are metrisable infinite-dimensional manifolds, see [Mic80]. Due to a theorem of Whitehead, see [Whi50], such spaces have the homotopy type of a CW-complex, albeit possibly a different one. In [Mil59], Milnor showed among other things that the space of maps from a finite CW-complex to any CW-complex is homotopy equivalent to a CW-complex. Therefore, taking the loop space does not tamper with the CW-structure. After realising the connecting map by a map on spaces via the ambient isotopy theorem, Whitehead's theorem gives the desired upgrade to a homotopy equivalence.

The setting of Theorem A is the one with dual spheres. Recall, let M be a d-dimensional, compact, oriented, connected manifold with non-empty boundary ∂M and consider a link $\ell : \mathbb{S}^{k-1} \hookrightarrow \partial M$ with a framed dual link $G : \mathbb{S}^{d-k,n} \hookrightarrow \partial M$. That is, we require that $\ell_i \pitchfork G_i = \mathrm{pt}$, $\ell_i \cap G_j = \emptyset$ for $i \neq j$ and G_i has trivialised normal bundle. We require that the link ℓ is slice in M, meaning the embedding space $\mathrm{Emb}_\ell(\mathbb{D}^{2,n}, M)$ is non-empty to begin with. To provide as many perspectives as possible, we give an alternative description in the case of $d \geq 2k$.

Proposition 12. *Let $d \geq 2k$. In the setting with dual spheres, the space $\mathrm{Emb}_\ell(\mathbb{D}^{k,n}, M)$ of neat embeddings is non-empty if and only if ℓ_i is null-homotopic in M.*

Proof. One direction is immediate. If the embedding space $\mathrm{Emb}_\ell(\mathbb{D}^{k,n}, M)$ is non-empty, there exists an embedded multi-disk $\mathbb{D}^{k,n}$ in M that bounds the link ℓ. Thus, every link component is null-homotopic by the corresponding component of the embedded multi-disk. Now, suppose that each component ℓ_i is null-homotopic in M. It is a well-known fact that a continuous map between smooth manifolds is homotopic to a smooth map, see for example [BT82, Proposition 17.8]. Therefore, we can assume that every null-homotopy is given by a smooth disk in M. Since the set of *generic immersions* is open and dense in $C^\infty(\mathbb{D}^{2,n}, M)$ with the Whitney C^∞-topology ([GG73, Chapter 3, Corollary 3.3]), we can furthermore assume that the set of null-homotopies is given by a generically immersed multi-disk. Hence, there are only finitely many points of transverse intersection. This is still far from being an embedding, as each disk can intersect itself transversally, and different disks can intersect each other transversally as well. We use the existence of the *framed* dual spheres to tube the disks away from each intersection point. This is known as Norman's trick, as discussed in [Nor69]. A dimensionally reduced version is depicted below in Figure 4. Note that there is a choice of the dual sphere we use to tube away a section of the multi-disk. Since the dual spheres are framed, we can tube away as many points of intersection as we want. Therefore, there exists an embedded multi-disk, and the space $\mathrm{Emb}_\ell(\mathbb{D}^{k,n}, M)$ is non-empty. □

To acquire Theorem A from Theorem 10, the final step lies in establishing a homotopy equivalence $\mathrm{Emb}_{\ell^\varepsilon}(\mathbb{D}^{k,n}, M) \simeq \mathrm{Emb}_{\ell^\varepsilon}(\mathbb{C}^{k,n}, M_G)$ where

$$M_G := M \cup_{\nu G} h^{d-k+1,n}$$

is being obtained by attaching a $(d-k+1)$-handle along the normal neighbourhood of each dual sphere G_i. In the setting of $n = 1$, this is the content of [KT23b, Section 3.2], and the argument in the case of multi-disks is practically the same. We follow the argument, although we only give a sketch in favour of readability. The idea is that attaching a multi-handle $h^{d-k+1,n}$ to the dual link is inverse to "drilling" out a neighbourhood of $u_+ = \mathrm{U}(\mathbb{D}_+^{k-1,n})$, which is removing a multi-handle $h^{k-1,n}$.

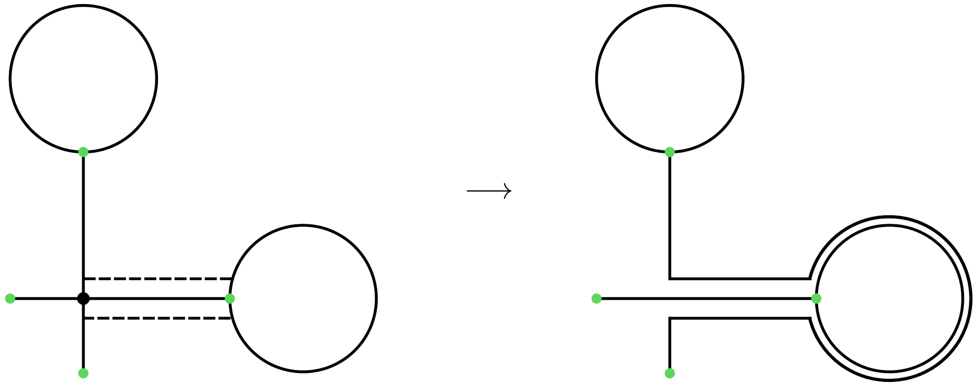


FIGURE 4. Performing a Norman trick to get rid of the intersection point p of the disks $\mathbb{D}_i^k$ and $\mathbb{D}_j^k$ in M.

We extend the embedding $u_+ : \mathbb{D}_+^{k-1,n} \hookrightarrow M_G$ to an embedding $\mathscr{T}u_+ : \mathbb{D}_+^{k-1,n} \times \mathbb{D}_{\leq\varepsilon}^{d-k+1} \hookrightarrow M_G$ onto a closed tubular neighbourhood $\overline{\nu}^\varepsilon u_+$ of u_+. By re-parametrisation, we may assume that the restriction of $\mathscr{T}u_+$ on $\mathbb{D}_+^{k-1,n} \times [0, \varepsilon]$ agrees with the ε-augmentation $u^\varepsilon = \mathrm{U}(\mathbb{D}_+^{\varepsilon,k-1,n})$ and that $\mathrm{im}(u_+^\varepsilon) = \overline{\nu}^\varepsilon u_+ \cap \mathrm{im}(\mathrm{U})$ by decreasing ε until the multi-half-disk U does not return to the closed neighbourhood $\overline{\nu}^\varepsilon u_+$ of u_+. In this setting, $\mathscr{T}u_+(\mathbb{D}_+^{k-1,n} \times \mathbb{D}_{\leq\varepsilon/2}^{d-k+1})$ is a multi-handle $h^{d-k+1,n}$ that is attached to the complement $M_G \setminus \mathscr{T}u_+(\mathbb{D}_+^{k-1,n} \times \mathbb{D}_{<\varepsilon/2}^{d-k+1})$. The other way around, the aforementioned complement is obtained by removing a $(k-1)$-multi-handle from M_G, where each component of the multi-handle has one component of $u_+(\mathbb{D}_+^{k-1,n})$ as core. The result is a manifold with corners which can be *smoothened*. This procedure is explained in [KT23b, Section 3.2] and we omit the technicality of explicitly dealing with smoothing corners here. As part of the procedure, one chooses an open subset $h_+^n \subseteq \overline{\nu}^\varepsilon u_+$ which is the open tubular neighbourhood $\nu^{\varepsilon/2} u_+$ with corners smoothed *inside* $\overline{\nu}^\varepsilon u_+$. This way, $M_G \setminus h_+^n$ is a smooth manifold with boundary. Note that removing h_+^n from M_G turns a multi-half-disk in M_G into a neat disk in $M_G \setminus h_+^n$. Let $\mathscr{D}^{k,n} := \mathscr{T}u_+^{-1}(h_+^n) \subseteq \text{ᗡ}^{k,n}$ be the multi-disk obtained in this way, before embedding it into $M_G \setminus h_+^n$. Let us fix an diffeomorphism $\mathbb{D}^{k,n} \cong \mathscr{D}^{k,n}$ which is the identity on a neighbourhood of $\mathbb{D}_-^{\varepsilon,k,n} \setminus \mathbb{D}_+^{\varepsilon,k,n}$, the set-wise difference of the outer boundary ε-neighbourhood and the inner boundary ε-neighbourhood.

Lemma 13. *There is a homotopy equivalence*

$$\mathrm{Emb}_{\ell^{\varepsilon/2}}(\mathscr{D}^{k,n}, M_G \setminus h_+^n) \xrightarrow{\bullet \cup u_+^\varepsilon} \mathrm{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, M_G).$$

The boundary condition $\ell^{\varepsilon/2}$ is needed such that this map is well-defined. Of course, one can scale ε accordingly and this does not change the homotopy type as it is just local re-parametrisation. The above lemma is precisely [KT23b, Lemma 3.9] in the case of multi-disks, but the proof can be repeated word by word. As it is technical, we omit it for the sake of taking load off of the notation.

At this point, the reader has most likely already seen why Lemma 13 is useful. Namely, since $M_G := M \cup_{\nu G} h^{d-k+1,n}$, removing a neighbourhood h^n_+ of the co-core of the attached multi-handle $h^{d-k+1,n}$ clearly gives back the original manifold M. Therefore, after possibly rescaling ε, we obtain the desired homotopy equivalence

$$\operatorname{Emb}_{\ell^\varepsilon}(\mathbb{D}^{k,n}, M) \simeq \operatorname{Emb}_{\ell^\varepsilon}(\mathbb{D}^{k,n}, M_G \setminus h^n_+) \simeq \operatorname{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, M_G).$$

Proof of Theorem A. The just mentioned homotopy equivalence combined with Theorem 10 applied to $X := M_G$ yields a proof of Theorem A. Indeed, we have

$$\operatorname{Emb}_{\ell^\varepsilon}(\mathbb{D}^{k,n}, M) \xrightarrow{\simeq} \operatorname{Emb}_{\ell^\varepsilon}(\text{ᗡ}^{k,n}, M_G) \xrightarrow[\simeq]{\text{Theorem } 10} \Omega \operatorname{Emb}^{\varepsilon}_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X)$$

and Lemma 9 yields a weak equivalence $\operatorname{Emb}_\ell(\mathbb{D}^{k,n}, M) \simeq_w \Omega \operatorname{Emb}^{\varepsilon}_{u_0^\varepsilon}(\mathbb{D}^{k-1,n}, X)$. The study of isotopy classes of multi-disks is a question about π_0 of embedding spaces, hence the weak equivalence suffices.

Remark 14. There is an obvious question regarding the difference of the homotopy type of ε-augmented embedding spaces appearing in Theorem A and standard embedding spaces. For the moment, we treat the former to be understood from the latter. A discussion will follow much later in Section V. If the reader is familiar with Hirsch-Smale theory, which is being recalled in Section IV, they are invited to skip ahead to complete the discussion on the homotopical reduction from $\operatorname{Emb}_\ell(\mathbb{D}^{2,n}, M)$ to $\operatorname{Emb}_{u_0}(\mathbb{D}^{1,n}, M_G)$.

On the history of half-disk tricks. As we mentioned in the proof of Theorem 10, the application of Theorem 4 to half-disks, or multi-half-disks in our case, is known as Cerf's half-disk trick which made its first appearance in [Cer62]. The half-disk trick was used to show the homotopy equivalence $\operatorname{Diff}_\partial(\mathbb{D}^d) \simeq \Omega \operatorname{Emb}_\partial(\mathbb{D}^{d-1}, \mathbb{D}^d)$. Note that Theorem A combined with Lemma 9 recovers this result, see also [KT23b]. Not only does it recover the result, but also its proof. Indeed, let $M = \mathbb{D}^{d,2}$, and consider the boundary condition $\ell = \partial \mathbb{D}^d_1$. The dual sphere $G : \mathbb{S}^0 \hookrightarrow \partial(\mathbb{D}^{d,2})$ has one point in $\partial \mathbb{D}^d_1$, another in $\partial \mathbb{D}^d_2$. In this case, $M_G \cong \mathbb{D}^d$ and we obtain

$$\operatorname{Diff}_\partial(\mathbb{D}^d) \cong \operatorname{Emb}_\ell(\mathbb{D}^d, \mathbb{D}^{d,2}) \simeq \Omega \operatorname{Emb}_{u_0}(\mathbb{D}^{d-1}, \mathbb{D}^d).$$

In his thesis [Goo90], Goodwillie studied smooth concordance embeddings. Let M be a smooth, d-dimensional manifold and $N \hookrightarrow M$ a fixed neat embedding of a compact manifold N. One should think about N as a submanifold of M. A *concordance embedding* of N into M is an embedding $\mathscr{E} : N \times \mathbb{I} \hookrightarrow M \times \mathbb{I}$ such that $\mathscr{E}^{-1}(M \times \{i\}) = N \times \{i\}$ for $i \in \{0,1\}$ and $\mathscr{E}$ agrees with the standard inclusion on a *neighbourhood* of $N \times \{0\} \cup (\partial M \cap N) \times \mathbb{I} \subseteq N \times \mathbb{I}$. Let $\mathsf{CE}(N, M)$ be the space of such embeddings, topologised by the smooth topology. Goodwillie used a half-disk trick to show the homotopy equivalence $\mathsf{CE}(\mathbb{D}^k, M \cup h^{d-k}) \simeq \Omega \mathsf{CE}(\mathbb{D}^{k-1}, M)$. A similarly flavoured delooping trick is discussed in [GKK23, Section 2.6] by Goodwillie, Krannich, and Kupers.

In [KK24], Knudsen and Kupers studied the dependence of the Goodwillie-Weiss tower on the smooth structure of both the source and target manifold. In particular, they showed that embedding calculus does not distinguish exotic smooth structures in dimension 4. The slogan seems to be that embedding calculus operates as if smoothing theory were true. During the discussion of an example of convergence in handle codimension 2, see [KK24, Section 6.2.4], they used the half-disk trick as presented in the proof of Theorem 10.

III. APPLICATION OF THE WORK OF DAX

The idea is to study the homotopy type of the embedding space $\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$ of neat embeddings of multi-arcs into a compact, oriented, connected 4-manifold M with non-empty boundary ∂M, by comparing it to the space of immersions, $\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M)$. There exist many strong tools to study the homotopy type of immersion spaces, mainly Hirsch-Smale theory. In the associated long exact sequence of homotopy groups of the pair $(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M))$, there is a relative homotopy group. In particular, we are interested in the second relative homotopy group $\pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$ for a chosen basepoint $u \in \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$. This agrees with the second homotopy group of the layer of the map $T_2 \longrightarrow T_1$ in the Goodwillie-Weiss tower associated to the embedding space $\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$. In this section, we compute this group using cobordism groups and an isomorphism introduced by Dax in [Dax72]. We give a geometric interpretation of the Dax isomorphism, together with an explicit inverse, generalising results by Kosanović and Teichner in [KT23b] to the setting of multi-arcs.

III.1. THE DAX ISOMORPHISM FOR MULTI-ARCS IN 4-MANIFOLDS

We begin with the statement of the Dax isomorphism and the definition of the cobordism groups introduced by Dax in our case of neat embeddings of multi-arcs in a 4-manifold. Let V be a smooth, compact r-manifold with non-empty boundary ∂V and X a smooth d-manifold with non-empty boundary ∂X. Let us fix a neat embedding $u : V \hookrightarrow X$.

Theorem 15 ([Dax72, p. 375])**.** *In this setting, if $d - 2r \geq 0$, then for $0 \leq n \leq 2d - 3r - 3$, there is an explicit isomorphism*

$$\pi_n(\mathrm{Imm}_\partial(V, X), \mathrm{Emb}_\partial(V, X), u) \xrightarrow{\beta_n} \Omega_{n-(d-2r)}(\mathcal{C}_u; \vartheta_u).$$

Here, we are considering normal cobordism classes of $n-(d-r)$-dimensional manifolds over a space $\mathcal{C}_u$ with a stable vector bundle ϑ_u. This includes the data of an $n-(d-r)$-dimensional closed manifold N, a continuous map $b : N \longrightarrow \mathcal{C}_u$, and a bundle isomorphism $\mathscr{B} : b^*(\vartheta_u) \longrightarrow \nu_N$, where ν_N denotes the stable normal bundle of N. We write this as a tuple $(N, b, \mathscr{B})$. A normal cobordism over $(N_1, b_1, \mathscr{B}_1)$ and $(N_2, b_2, \mathscr{B}_2)$ consists of the following data. A compact manifold W with $\partial W = N_1 + N_2$, an extension g of $b_1 + b_2$ to W such that the following diagram commutes

$$\begin{tikzcd}
N_1 \arrow[d, "\iota_2"'] \arrow[dr, "b_1"] & \\
W \arrow[r, "g"] & \mathcal{C}_u \\
N_2 \arrow[u, "\iota_1"] \arrow[ur, "b_2"'] &
\end{tikzcd}$$

and an extension of $\mathscr{B}_1 + \mathscr{B}_2$ to an isomorphism $\mathscr{G} : g^*(\vartheta_u) \longrightarrow \nu_W$. On the boundary of W, we identify $\nu_W|_{\partial W}$ with $\nu_{\partial W} \oplus \mathbb{R}$ by an inward pointing vector field.

Remark 16. In the case of the space we take cobordism classes over (in our notation $\mathcal{C}_u$) is a point, we obtain *framed* cobordism classes Ω_n^{fr}. The Pontrjagin collapse map yields an isomorphism

$$\Omega_n^{\mathrm{fr}} \xrightarrow{\cong} \pi_n(\mathbb{S}) := \varinjlim_k \pi_{n+k}(\mathbb{S}^k)$$

to the n-th homotopy group of the sphere spectrum. This is one of the major connections between differential geometry and stable homotopy theory.

Following the spirit of Kosanović and Teichner in [KT23b, Section 4.1], we do not give a full description of the cobordism group as originally defined by Dax, but state the needed properties to understand and work with the Dax isomorphism. Another introduction to the work of Dax, together with a comparison to the work of Haefliger, can be found in [GKW01, Chapter 1]. Before we direct our attention to the definition of the cobordism group, we recall the notion of ordered and unordered configuration spaces.

Definition 17. Let X be a space and $k \in \mathbb{N}$. The *ordered configuration space* of k points in X is defined as $\widetilde{\mathsf{Conf}}_k(X) := \mathrm{Emb}((x_1, \ldots, x_k), X)$. The symmetric group S_k acts on this space by renumbering the points. Forming the quotient, the base space $\mathsf{Conf}_k(X)$ of the fibration sequence

$$S_k \longrightarrow \widetilde{\mathsf{Conf}}_k(X) \longrightarrow \mathsf{Conf}_k(X)$$

is called the *unordered configuration space* of k points in X. It is evident that

$$\widetilde{\mathsf{Conf}}_k(X) \cong X^k \setminus \{(x_1, \ldots, x_k) \in X^k : x_i = x_j \text{ for some } i \neq j\}$$

which is the identification we will use.

Back to the Dax isomorphism, we define a homotopy equivalent subspace of $\mathcal{C}_u$.

Definition 18. – There is a subspace $E_u^{\curlyvee} \subseteq \mathcal{C}_u$ which is given by the quotient of the space

$$\widetilde{E}_u^{\curlyvee} := \{(v_1, v_2, \gamma) \in \widetilde{\mathsf{Conf}}_2(V) \times \mathrm{Map}([-1,1], X) : \gamma(-1) = u(v_1) \text{ and } \gamma(1) = u(v_2)\}$$

by the $\mathbb{Z}/2$-action given by the free involution sending (v_1, v_2, γ) to (v_2, v_1, γ^{-1}). This means that, in $E_u^{\curlyvee}$, paths from $u(v_1)$ to $u(v_2)$ are identified with their inverse paths from $u(v_2)$ to $u(v_1)$. Classes in $E_u^{\curlyvee}$ are denoted by $[v_1, v_2, \gamma]$.

- Let ν_V be the stable normal bundle over V, TX the tangent bundle of X. Let $\widetilde{\vartheta}_u$ be the bundle over $\widetilde{E}_u^{\Upsilon}$ defined via the following pullback diagram.

$$\begin{tikzcd}
\widetilde{\vartheta}_u \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \nu_V^2 \oplus TX \arrow[d] \\
\widetilde{E}_u^{\Upsilon} \arrow[r, "(\mathsf{pr}_{V^2}\text{,}\mathsf{pr}_0)"] & V^2 \times X
\end{tikzcd}$$

  The map $(\mathsf{pr}_{V^2}, \mathsf{pr}_0) : \widetilde{E}_u^{\Upsilon} \longrightarrow V^2 \times X$ sends (v_1, v_2, γ) to $(v_1, v_2, \gamma(0))$ and is equivariant for the $\mathbb{Z}/2$-action given by the involution swapping the two coordinates of V^2 and the identity on X. The quotient of $\widetilde{\vartheta}_u$ by this $\mathbb{Z}/2$-action is the restriction bundle $\vartheta_u|_{E_u^{\Upsilon}}$.

Remark 19. Since $\nu_X \oplus TX$ is trivial per definition, and $\nu_V \cong \nu_u \oplus \nu_X$, there is an isomorphism $\widetilde{\vartheta}_u := \mathsf{pr}_{V^2}^*(\nu_V^2) \oplus \mathsf{pr}_X^*(TX) \cong \mathsf{pr}_{V^2}^*(\nu_u^2) \oplus \mathsf{pr}_X^*(\nu_X)$.

Lemma 20. *The subspace E_u^{Υ} is homotopy equivalent to $\mathcal{C}_u$.*

Proof. This is discussed in [KT23b, Lemma 4.5], using a fibration sequence $\Omega X \longrightarrow \mathcal{C}_u \xrightarrow{\mathsf{pr}_W} W$, where W is the compactification to a manifold with boundary of $\mathsf{Conf}_2(V)$. The subspace E_u^{Υ} can be identified with $\mathsf{pr}_W^{-1}(\dot{W})$. The inclusion map $i : \mathsf{Conf}_2(V) = \dot{W} \hookrightarrow W$ is a homotopy equivalence, hence by the pullback diagram

$$\begin{tikzcd}
\mathsf{pr}_W^{-1}(\dot{W}) \arrow[r, equal] & i^*(\mathcal{C}_u) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}_u \arrow[d, "\mathsf{pr}_W"] \\
& \dot{W} \arrow[r, "i"] & W
\end{tikzcd}$$

there is a homotopy equivalence $E_u^{\Upsilon} \simeq \mathcal{C}_u$. □

The above lemma allows us to consider the simpler cobordism group $\Omega_{n-(d-2r)}(E_u^{\Upsilon}; \vartheta_u|_{E_u^{\Upsilon}})$ instead of the original one $\Omega_{n-(d-2r)}(\mathcal{C}_u; \vartheta_u)$ leading to an isomorphism

$$\pi_n(\mathrm{Imm}_\partial(V, X), \mathrm{Emb}_\partial(V, X), u) \xrightarrow{\beta_n'} \Omega_{n-(d-2r)}(E_u^{\Upsilon}; \vartheta_u|_{E_u^{\Upsilon}}).$$

We now tend to the case of neat multi-arcs in a compact, oriented, connected 4-manifold M with non-empty boundary ∂M. Let $u \in \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$ be a chosen basepoint. As just discussed, Theorem 15 combined with Lemma 20 yields an isomorphism

$$\beta_2' : \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \longrightarrow \Omega_0(E_u^{\Upsilon}; \vartheta_u|_{E_u^{\Upsilon}})$$

which we now explicitly describe following [KT23b], to which we refer the reader for the discussion of the general case.

Remark 21. This discussion should be compared to the second stage in the Goodwillie-Weiss tower

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \mathsf{T}_2\,\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M) \arrow[d] \\
\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M) \arrow[r, "\mathsf{ev}_1"] \arrow[ur, "\mathsf{ev}_2"] \arrow[uur, bend left] & \mathsf{T}_1\,\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)
\end{tikzcd}$$

for the space $\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$. Note that the first stage $\mathsf{T}_1\,\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$ is homotopy equivalent to $\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M)$. Indeed, the fundamental theorem of embedding calculus, see for example [Wei99] and [GW99], implies that the evaluation map

$$\mathsf{ev}_2 : \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M) \longrightarrow \mathsf{T}_2\,\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$$

to the second stage in the tower is 2-connected. A careful introduction can be found in [BW13] or [BW18] for a more modern approach studying functors out of a configuration ∞-category. For the sake of less cluttered notation, we now abbreviate $\mathrm{Emb}_\partial := \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$, $\mathrm{Imm}_\partial := \mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M))$ and $\mathsf{T}_k := \mathsf{T}_k\,\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$. Comparing the long exact sequences of homotopy groups associated to the pairs $(\mathrm{Imm}_\partial, \mathrm{Emb}_\partial)$ and $(\mathsf{T}_1, \mathsf{T}_2)$ yields the following commutative diagram.

$$\begin{tikzcd}
\pi_2(\mathrm{Emb}_\partial, u) \arrow[r] \arrow[d, two heads, "\pi_2(\mathsf{ev}_2)"] & \pi_2(\mathrm{Imm}_\partial, u) \arrow[r] \arrow[d, tail, two heads] & \pi_2(\mathrm{Imm}_\partial, \mathrm{Emb}_\partial, u) \arrow[r, "\partial"] \arrow[d, "\pi_2(\mathsf{ev}_2)"] & \pi_1(\mathrm{Emb}_\partial, u) \arrow[r] \arrow[d, tail, two heads, "\pi_1(\mathsf{ev}_2)"] & \pi_1(\mathrm{Imm}_\partial, u) \arrow[d, tail, two heads] \\
\pi_2(\mathsf{T}_2, u) \arrow[r] & \pi_2(\mathsf{T}_1, u) \arrow[r] & \pi_2(\mathsf{T}_1, \mathsf{T}_2, u) \arrow[r, "\partial"] & \pi_1(\mathsf{T}_2, u) \arrow[r] & \pi_1(\mathsf{T}_1, u)
\end{tikzcd}$$

Applying the 5-Lemma yields an isomorphism

$$\pi_2(\mathsf{ev}_2) : \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \xrightarrow{\cong} \pi_2(\mathsf{T}_1, \mathsf{T}_2, u)$$

of relative homotopy groups. Therefore, the Dax isomorphism β_2' computes the second homotopy group of the layer associated to the map $\mathsf{T}_2 \longrightarrow \mathsf{T}_1$ in the Goodwillie-Weiss tower. A natural question to ask is whether one can, in a stable range, express the homotopy groups of the higher layers in the tower as certain cobordism groups, computable via the Atiyah-Hirzebruch spectral sequence.

Back to the definition of β_2', an element in $\pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}), M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$ is given by the homotopy class of a map

$$F : (\mathbb{I}^2, \mathbb{I} \times \{0\}, \partial\mathbb{I} \times \mathbb{I} \cup \mathbb{I} \times \{1\}) \longrightarrow (\mathrm{Imm}_\partial(\mathbb{D}^{1,n}), M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$$

of pointed pairs of spaces. One should think about such a map in the following pictorial way for $n = 1$, scanning through the square $\mathbb{I}^2$. This can be seen in Figure 5.

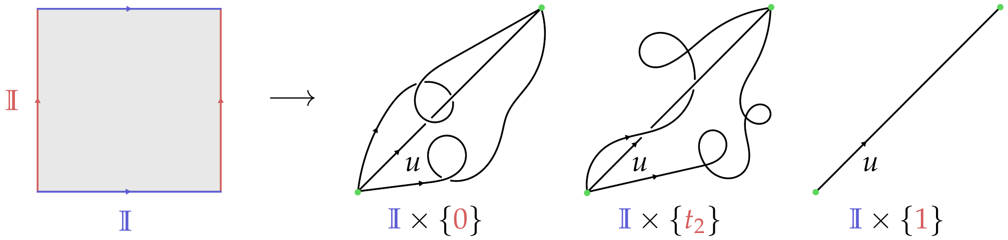


FIGURE 5. Scanning through $\mathbb{I} \times \{0\}$ starts and ends with the embedding u, embeddings $\mathbb{D}^1 \hookrightarrow M$ in between. Scanning through $\mathbb{I} \times \{t_2\}$ for $0 < t_2 < 1$ starts and ends with the embeddings u, immersions $\mathbb{D}^1 \looparrowright M$ in between. Scanning through $\mathbb{I} \times \{1\}$ is constantly the embedding u.

Note that, by adjunction for the Whitney C^∞-topology, this can be seen as a map $\mathbb{I}^2 \times \mathbb{D}^{1,n} \longrightarrow M$ with the obvious boundary conditions. Evaluating the map $F(t_1, t_2)$ on $x \in \mathbb{D}^{1,n}$ yields the associated track

$$\widetilde{F} : \underbrace{\mathbb{I}^2 \times \mathbb{D}^{1,n}}_{\text{dimension 3}} \longrightarrow \underbrace{\mathbb{I}^2 \times M}_{\text{dimension 6}} \text{ by sending } (t_1, t_2, x) \text{ to } (t_1, t_2, F(t_1, t_2)(x))$$

which, by the work of Dax in [Dax72, p. 329] (see [Dax72, p. 331] for the relative case), we can assume to be an immersion without triple points, and only finitely many double points that are isolated. Such a map F, whose track satisfies these conditions, is called *perfect*. We will now explicitly describe the image of $[F]$ under the isomorphism β_2', the cobordism class of the tuple $(\Delta_{\mathrm{Dax}}, b_{\mathrm{Dax}}, \mathscr{B}_{\mathrm{Dax}})$. Note that F being perfect is equivalent to the hollow diagonal square map

$$\widetilde{F}_h^2 : \widetilde{\mathsf{Conf}}_2(\mathbb{I}^2 \times \mathbb{D}^{1,n}) \longrightarrow (\mathbb{I}^2 \times M)^2$$

being *transverse* to the diagonal $\Delta_{\mathbb{I}^2 \times M}$ in $(\mathbb{I}^2 \times M)^2$. Let us assume that the double points of the track $\widetilde{F}$ are given by $p_1, \ldots, p_m$. The double point pre-image set is given by

$$\widetilde{\Delta}_{\mathrm{Dax}} := (\widetilde{F}_h^2)^{-1}(\Delta_{\mathbb{I}^2 \times M}) \cong \{(\vec{t}, x, y) \in \mathbb{I}^2 \times \widetilde{\mathsf{Conf}}_2(\mathbb{D}^{1,n}) : F(\vec{t})(x) = F(\vec{t})(y)\}$$

with $\vec{t} = (t_1, t_2) \in \mathbb{I}^2$. Note that the pre-image points in this case are *ordered*. Forming the quotient by the $\mathbb{Z}/2$-action given by the free involution interchanging the two coordinates yields the space

$$\Delta_{\mathrm{Dax}} := \{(\vec{t}, [x, y]) \in \mathbb{I}^2 \times \mathsf{Conf}_2(\mathbb{D}^{1,n}) : F(\vec{t})(x) = F(\vec{t})(y)\}.$$

Clearly, the map F embeds Δ_{Dax} into the manifold M as the double point set of the track $\widetilde{F}$. We now define the map $b_{\mathrm{Dax}} : \Delta_{\mathrm{Dax}} \longrightarrow E_u^\curlyvee$ as follows. For $\vec{t} \in \mathbb{I}^2$, let $\mathscr{H}_{\vec{t}} : \mathbb{I} \longrightarrow \mathbb{I}^2$ be defined as $\mathscr{H}_{\vec{t}}(s) := \vec{1} - s(\vec{1} - \vec{t})$, the linear path from $\vec{1} = (1, 1)$ to $\vec{t} = (t_1, t_2)$. The map

$$b_{\mathrm{Dax}}(\vec{t}, [x, y]) := [x, y, \gamma := F(\mathscr{H}_{\vec{t}})(x) \cdot F(\mathscr{H}_{\vec{t}})(y)^{-1}]$$

sends an element $(\vec{t}, [x, y]) \in \Delta_{\mathrm{Dax}}$ to the tuple $[x, y]$ together with a loop from $u(x)$ to $u(y)$. Ignoring the evaluation at x and y for a moment, $F(\mathscr{H}_{\vec{t}})$ is a path from $F(\vec{1}) = u$ to $F(\vec{t})$ through immersions. We start by evaluating each immersion at x, starting from $u(x)$ and ending at $F(\vec{t})(x)$. Since $(\vec{t},[x,y]) \in \Delta_{\mathrm{Dax}}$, we know that $F(\vec{t})(x)$ and $F(\vec{t})(y)$ agree as a double point p. We then return to $u(y)$ by tracing back through the immersions, evaluating at y. Therefore this indeed defines a path from $u(x)$ to $u(y)$. We need to distinguish two fundamentally different kinds of double points p, as this will become important in the upcoming section.

- Those, whose pre-image points $\{(\vec{t},x),(\vec{t},y)\} = \tilde{F}^{-1}(\{\vec{t},p\})$ lie in the same connected component of $\mathbb{I}^2 \times \mathbb{D}^{1,n}$, which is depicted on the left side in Figure 6.
- Those, whose pre-image points $\{(\vec{t},x),(\vec{t},y)\} = \tilde{F}^{-1}(\{\vec{t},p\})$ lie in different connected components of $\mathbb{I}^2 \times \mathbb{D}^{1,n}$, which is depicted on the right side in Figure 6.

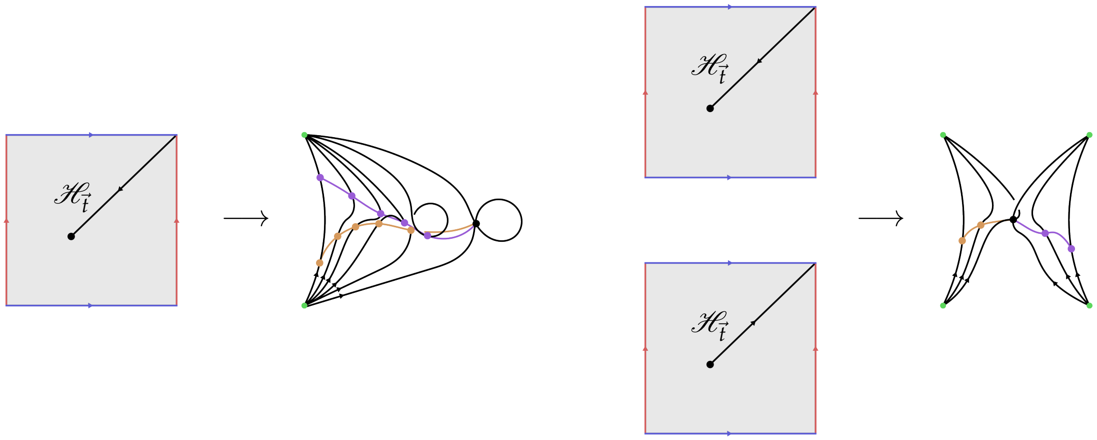


FIGURE 6. In both cases, the paths are given by following the points. The path corresponding to self-intersection of a single disk on the left-hand side. Note that the points follow the movement of the "twisting" motion, tracing out a loop. This is better seen when thinking about a *movie* of twisting the arc, where each state is displayed by one of the arcs in the figure. The path corresponding to intersection of different sheets on the right-hand side.

To define the cobordism class $(\Delta_{\mathrm{Dax}}, b_{\mathrm{Dax}}, \mathscr{B}_{\mathrm{Dax}}) \in \Omega_0(E_u^{\curlyvee}; \vartheta_u|_{E_u^{\curlyvee}})$ we are only missing information coming from the stable normal bundle in form of $\mathscr{B}_{\mathrm{Dax}}$, an isomorphism from $b_{\mathrm{Dax}}^*(\vartheta_u|_{E_u^{\curlyvee}})$ to the stable normal bundle $\nu_{\Delta_{\mathrm{Dax}}}$.

$$\begin{tikzcd}
\nu_{\Delta_{\mathrm{Dax}}} \arrow[dr] & b_{\mathrm{Dax}}^*(\vartheta_u|_{E_u^{\curlyvee}}) \arrow[l, "\mathscr{B}_{\mathrm{Dax}}"', "\cong"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \vartheta_u|_{E_u^{\curlyvee}} \arrow[d] \\
 & \Delta_{\mathrm{Dax}} \arrow[r, "b_{\mathrm{Dax}}"] & E_u^{\curlyvee}
\end{tikzcd}$$

Consider the quotient map $q : \widetilde{\Delta}_{\mathrm{Dax}} \longrightarrow \Delta_{\mathrm{Dax}}$ given by the involution. For the pullback bundle $q^*\nu_{\Delta_{\mathrm{Dax}}}$, we have a stable isomorphism

$$q^*\nu_{\Delta_{\mathrm{Dax}}} \cong \nu_{\widetilde{\Delta}_{\mathrm{Dax}}} \cong \nu_{(\mathbb{I}^2\times\mathbb{D}^{1,n})^2}|_{\widetilde{\Delta}_{\mathrm{Dax}}} \oplus \nu_{\widetilde{\Delta}_{\mathrm{Dax}}\subseteq(\mathbb{I}^2\times\mathbb{D}^{1,n})^2} \cong_s \nu^2_{\mathbb{D}^{1,n}}|_{\widetilde{\Delta}_{\mathrm{Dax}}} \oplus (\widetilde{F}_h^2)^*(\nu_{\Delta_{\mathbb{I}^2\times M}\subseteq(\mathbb{I}^2\times M)^2})$$

by tracing through the definition of $\widetilde{\Delta}_{\mathrm{Dax}}$. In particular, this yields a stable isomorphism

$$\nu_{\widetilde{\Delta}_{\mathrm{Dax}} \subseteq (\mathbb{I}^2 \times \mathbb{D}^{1,n})^2} \cong_s (\widetilde{F}^2_h)^*(\nu_{\Delta_{\mathbb{I}^2 \times M} \subseteq (\mathbb{I}^2 \times M)^2}).$$

For $b_{\mathrm{Dax}} : \Delta_{\mathrm{Dax}} \longrightarrow E_u^{\Upsilon}$, there is a double covering map $\widetilde{b}_{\mathrm{Dax}} : \widetilde{\Delta}_{\mathrm{Dax}} \longrightarrow \widetilde{E}_u^{\Upsilon}$ which induces an isomorphism $q^* b^*_{\mathrm{Dax}}(\vartheta_u|_{E_u^{\Upsilon}}) \cong \widetilde{b}^*_{\mathrm{Dax}}(\widetilde{\vartheta}_u)$. We first establish a stable bundle isomorphism $\widetilde{\mathscr{B}}_{\mathrm{Dax}} : \widetilde{b}^*_{\mathrm{Dax}}(\widetilde{\vartheta}_u) \longrightarrow q^* \nu_{\Delta_{\mathrm{Dax}}}$ by considering both components of $\widetilde{\vartheta}_u := \mathsf{pr}^*_{(\mathbb{D}^{1,n})^2} \nu^2_{\mathbb{D}^{1,n}} \oplus \mathsf{pr}^*_M(TM)$ (recall Definition 18). The first component is immediate, there is an isomorphism $\widetilde{b}^*_{\mathrm{Dax}}(\mathsf{pr}^*_{(\mathbb{D}^{1,n})^2} \nu^2_{\mathbb{D}^{1,n}}) \cong \nu^2_{\mathbb{D}^{1,n}}|_{\widetilde{\Delta}_{\mathrm{Dax}}}$ by definition. For the second component, consider the diagram

$$\begin{tikzcd}
\widetilde{\Delta}_{\mathrm{Dax}} \arrow[r, "\widetilde{b}_{\mathrm{Dax}}"] \arrow[d, "\widetilde{F}^2_h|_{\widetilde{\Delta}_{\mathrm{Dax}}}"'] & \widetilde{E}_u^{\Upsilon} \arrow[d, "\mathsf{pr}_M"] \\
(\mathbb{I}^2 \times M)^2 \arrow[r, "\mathsf{pr}_1"] & M
\end{tikzcd}$$

which commutes as evaluation on $(\vec{t}, x, y)$ yields both $F(\vec{t})(x)$ by definition of $\widetilde{\Delta}_{\mathrm{Dax}}$. This yields a stable bundle isomorphism

$$\widetilde{b}^*_{\mathrm{Dax}}(\mathsf{pr}^*_M(TM)) \cong (\mathsf{pr}_1 \circ \widetilde{F}^2_h)^*(TM) \cong_s (\widetilde{F}^2_h)^*(\mathsf{pr}^*_1(T(\mathbb{I}^2 \times M)) \cong (\widetilde{F}^2_h)^*(\nu_{\Delta_{\mathbb{I}^2 \times M} \subseteq (\mathbb{I}^2 \times M)^2}).$$

Hence, there is a stable bundle isomorphism $\widetilde{\mathscr{B}}_{\mathrm{Dax}} : \widetilde{b}^*_{\mathrm{Dax}}(\widetilde{\vartheta}_u) \cong q^* b^*_{\mathrm{Dax}}(\vartheta_u|_{E_u^{\Upsilon}}) \longrightarrow q^* \nu_{\Delta_{\mathrm{Dax}}}$ which is $\mathbb{Z}/2$-equivariant under the involution. Hence it descends to a stable bundle isomorphism $\mathscr{B}_{\mathrm{Dax}} : b^*_{\mathrm{Dax}}(\vartheta_u|_{E_u^{\Upsilon}}) \longrightarrow \nu_{\Delta_{\mathrm{Dax}}}$.

For a chosen basepoint $u \in \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$, the isomorphism

$$\beta'_2 : \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \longrightarrow \Omega_0(E_u^{\Upsilon}; \vartheta_u|_{E_u^{\Upsilon}})$$

provided by Theorem 15 and Lemma 20 is defined by sending a homotopy class $[F]$ to the cobordism class $(\Delta_{\mathrm{Dax}}, b_{\mathrm{Dax}}, \mathscr{B}_{\mathrm{Dax}}) \in \Omega_0(E_u^{\Upsilon}; \vartheta_u|_{E_u^{\Upsilon}})$.

III.2. A GEOMETRIC INTERPRETATION OF THE DAX ISOMORPHISM

Computing the cobordism group. Before we give a geometrically flavoured isomorphism, let us begin with computing the cobordism group in Theorem 15 in the case of multi-arcs in a 4-manifold. This is the content of the following theorem.

Theorem 22. *Let M be a compact, oriented, connected 4-manifold with non-empty boundary ∂M. For any choice of a basepoint $u \in \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$ of the space of neat embeddings, there is an isomorphism*

$$\Omega_0(E_u^{\Upsilon}; \vartheta_u|_{E_u^{\Upsilon}}) \cong \mathbb{Z}[\pi_0(\mathsf{Conf}_2(\mathbb{D}^{1,n}), b_0) \times \pi_1(M)]$$

and we identify $\pi_0(\mathsf{Conf}_2(\mathbb{D}^{1,n}), b_0)$ with $\mathbb{T}_n := \{(i_1, i_2) \in \{1, \ldots, n\}^2 : i_1 \leq i_2\}$.

Before we can give a proof of the above theorem, we need a technical well-known result on double-evaluation maps.

Lemma 23. *Let X be a locally compact Hausdorff space, $i : A \hookrightarrow X$ be a closed cofibration. For any space Y, the map*

$$i^* : \mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(A, Y)$$

induced by pre-composition is a fibration.

Proof. Let Z be some space. The lifting problem is the following.

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left, "H"] \arrow[ddr, bend right, "f"'] & & \\
& \mathrm{Map}(\mathbb{I}, \mathrm{Map}(X, Y)) \arrow[r] \arrow[d, "\mathsf{ev}_0"'] & \mathrm{Map}(\mathbb{I}, \mathrm{Map}(A, Y)) \arrow[d, "\mathsf{ev}_0"] \\
& \mathrm{Map}(X, Y) \arrow[r, "i^*"] & \mathrm{Map}(A, Y)
\end{tikzcd}$$

By adjunction, we can consider the following diagram.

$$\begin{tikzcd}
Z \times A \arrow[r, hook] \arrow[d, hook] & Z \times A \times \mathbb{I} \arrow[d, hook] \arrow[ddr, bend left, "H^\flat"] & \\
Z \times X \arrow[r, hook] \arrow[drr, bend right, "f^\flat"'] & Z \times X \times \mathbb{I} \arrow[dr, dashed] & \\
& & Y
\end{tikzcd}$$

Since i is a closed cofibration, there exists a map $Z \times X \times \mathbb{I} \longrightarrow Y$ making the second diagram commute. The adjoint to this map is the dotted arrow in the first diagram, thus a lift exists. □

This result is to be compared to Theorem 4. The situation of mapping spaces is much easier to handle from the perspective of homotopy theory.

Corollary 24. *The double-evaluation map* $(\mathsf{ev}_0, \mathsf{ev}_1) : \mathrm{Map}(\mathbb{I}, X) \longrightarrow X \times X$ *is a fibration.*

Proof. The inclusion $\partial \mathbb{I} \hookrightarrow \mathbb{I}$ is a closed cofibration. Now we apply Lemma 23. □

Equipped with this result, we can tackle the proof of Theorem 22.

Proof of Theorem 22. We first calculate $\pi_0(\widetilde{E}_u^{\Upsilon}, \widetilde{e}_0)$ for a chosen basepoint $\tilde{e}_0 \in \widetilde{E}_u^{\Upsilon}$. Notice that we have a pullback diagram

$$\begin{tikzcd}
\widetilde{E}_u^{\Upsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Map}(\mathbb{I}, M) \arrow[d, "(\mathsf{ev}_0{,}\mathsf{ev}_1)"] \\
\widetilde{\mathsf{Conf}}_2(\mathbb{D}^{1,n}) \arrow[r, "(u{,}u)"] & M \times M
\end{tikzcd}$$

which induces a long exact sequence of homotopy groups since the double-evaluation map $(\mathsf{ev}_0, \mathsf{ev}_1) : \mathrm{Map}(\mathbb{I}, M) \longrightarrow M \times M$ is a fibration by Corollary 24. Hence, the induced map $\widetilde{E}_u^{\Upsilon} \longrightarrow (\mathbb{D}^{1,n})^2 \setminus \Delta_{\mathbb{D}^{1,n}}$ is a fibration as well. Let $\widetilde{b}_0 \in \widetilde{\mathsf{Conf}}_2(\mathbb{D}^{1,n})$ be the image of the basepoint

$\widetilde{e}_0$ under this map, chosen as a canonical basepoint. We have the following long exact sequence of homotopy groups.

$$\begin{tikzcd}
\dots \arrow[r] & \pi_1(\widetilde{\mathsf{Conf}_2}(\mathbb{D}^{1,n}),\widetilde{b}_0)\times\pi_1(\mathsf{Map}(\mathbb{I},M)) \arrow[r, "(u,u)_*-(\mathsf{ev}_0{,}\mathsf{ev}_1)_*"] & \pi_1(M\times M) \arrow[dll, rounded corners] \\
\pi_0(\widetilde{E}^{\Upsilon}_u,\widetilde{e}_0) \arrow[r, two heads] & \pi_0(\widetilde{\mathsf{Conf}_2}(\mathbb{D}^{1,n}),\widetilde{b}_0)\times\pi_0(\mathsf{Map}(\mathbb{I},M)). &
\end{tikzcd}$$

Note that $\mathsf{Map}(\mathbb{I},M)$ is homotopy equivalent to M by contracting each interval to a chosen endpoint. Since M is assumed to be connected, we suppress the basepoint in the notation. Each connected component of $\widetilde{\mathsf{Conf}_2}(\mathbb{D}^{1,n})\cong(\mathbb{D}^{1,n})^2\setminus\Delta_{\mathbb{D}^{1,n}}$ is contractible, hence the pure braid group $\pi_1(\widetilde{\mathsf{Conf}_2}(\mathbb{D}^{1,n}),\widetilde{b}_0)$ is trivial. Last but not least, $(\mathsf{ev}_0,\mathsf{ev}_1)_*$ is the diagonal map, as for any $\gamma\in\pi_1(\mathsf{Map}(\mathbb{I},M))$, the induced elements $\mathsf{ev}_0(\gamma)$ and $\mathsf{ev}_1(\gamma)$ are homotopic through the interval $\mathbb{I}$. This leaves us with the following four-term exact sequence.

$$\begin{tikzcd}
\pi_1(M) \arrow[r, tail, "\Delta"] & \pi_1(M)\times\pi_1(M) \arrow[r] & \pi_0(\widetilde{E}^{\Upsilon}_u,\widetilde{e}_0) \arrow[r, two heads] & \pi_0(\widetilde{\mathsf{Conf}_2}(\mathbb{D}^{1,n}),\widetilde{b}_0)
\end{tikzcd}$$

and therefore $\pi_0(\widetilde{E}^{\Upsilon}_u,\widetilde{e}_0)\cong\pi_0(\widetilde{\mathsf{Conf}_2}(\mathbb{D}^{1,n}),\widetilde{b}_0)\times\pi_1(M)$. Now we consider the fibration sequence

$$\mathbb{Z}/2\longrightarrow\widetilde{E}^{\Upsilon}_u\longrightarrow E^{\Upsilon}_u$$

induced by the involution. Invoking the long exact sequence of homotopy groups immediately yields $\pi_0(E^{\Upsilon}_u,e_0)\cong\pi_0(\mathsf{Conf}_2(\mathbb{D}^{1,n}),b_0)\times\pi_1(M)$ for e_0 and b_0 the canonical basepoints of E^{Υ}_u and $\mathsf{Conf}_2(\mathbb{D}^{1,n})$ respectively. Note that the initial choice of $\widetilde{e}_0$ does not matter as no other homotopy group in the four-term exact sequence depends on a basepoint. The product space $(\mathbb{D}^{1,n})^2$ clearly has n^2 path-connected components. Removing the diagonal $\Delta_{\mathbb{D}^{1,n}}$ results in n additional path-connected components, thus $\pi_0(\widetilde{\mathsf{Conf}_2}(\mathbb{D}^{1,n}),b_0)\cong\{1,\dots,n^2+n\}$ as a set. The involution acts as a “folding map”, leaving exactly ${}^{n(n+1)}/_{2}$ path-connected components. Hence $\pi_0(\mathsf{Conf}_2(\mathbb{D}^{1,n}),b_0)\cong\mathbb{T}_n$. To compute the cobordism group $\Omega_0(E^{\Upsilon}_u;\vartheta_u|_{E^{\Upsilon}_u})$ we note that there is an isomorphism $\Omega_0(E^{\Upsilon}_u;\vartheta_u|_{E^{\Upsilon}_u})\cong H_0(E^{\Upsilon}_u;\mathbb{Z}(\vartheta_u|_{E^{\Upsilon}_u}))$ as we are considering cobordism of 0-dimensional manifolds over E^{Υ}_u. Here, $\mathbb{Z}(\vartheta_u|_{E^{\Upsilon}_u})$ denotes the local coefficient system induced by the orientation of the bundle $\vartheta_u|_{E^{\Upsilon}_u}$ over connected components of E^{Υ}_u. We show that this is constant and $\mathbb{Z}$. By the definition of the bundle $\vartheta_u|_{E^{\Upsilon}_u}$ and Remark 19, this amounts to showing that there cannot exist a loop in $\mathsf{Conf}_2(\mathbb{D}^{1,n})$ that lifts to E^{Υ}_u under $\mathsf{pr}_{\mathbb{D}^{1,n}}:E^{\Upsilon}_u\longrightarrow\mathsf{Conf}_2(\mathbb{D}^{1,n})$ and is orientation-reversing in the quotient of ν^2_u under the involution. This is given since $\pi_1(\mathsf{Conf}_2(\mathbb{D}^{1,n}),b_0)$ is trivial. Therefore, the bundle $\vartheta_u|_{E^{\Upsilon}_u}$ is orientable over each connected component, hence the local coefficient system is constant and $\mathbb{Z}$. This leaves us with the desired isomorphism $\Omega_0(E^{\Upsilon}_u;\vartheta_u|_{E^{\Upsilon}_u})\cong\mathbb{Z}[\mathbb{T}_n\times\pi_1(M)]$. □

Remark 25. One can similarly write $\mathbb{Z}[\mathbb{T}_n\times\pi_1(M)]\cong\bigoplus_{\mathbb{T}_n}\mathbb{Z}[\pi_1(M)]$ but our notation will come in handy later in Section IV.

Kosanović and Teichner have a version of Theorem 22 that allows the domain manifold to be any simply-connected manifold V and the target manifold M to be of arbitrary dimension, as long as the conditions of Theorem 15 are satisfied, see [KT23b, Proposition 4.12]. The above discussion immediately gives rise to the case of disjoint unions of simply-connected manifolds. If the domain manifold V is not simply-connected, one could obtain results by restricting to those manifolds that induce an injection

$$\pi_1(\widetilde{\mathsf{Conf}}_2(V), \widetilde{v}_0) \times \pi_1(\mathrm{Map}(\mathbb{I}, M)) \rightarrowtail \pi_1(M \times M).$$

This can be reduced to an injection $\pi_1(\mathsf{Conf}_2(V), v_0) \rightarrowtail \pi_1(M)$ from the braid group into $\pi_1(M)$ by considering the fibration sequence $E_u^{\Upsilon} \xrightarrow{\mathsf{pr}_{V2}} \mathsf{Conf}_2(V)$, see Lemma 27 for a discussion in the case of $V = \mathbb{D}^{1,n}$. In the case of V connected and $\dim(V) \geq 3$, the inclusion $\widetilde{\mathsf{Conf}}_k(V) \hookrightarrow V^k$ induces an isomorphism on fundamental groups. This follows from considering the fat diagonal as a union of submanifolds of M^k and using Thom's transversality theorem to homotope every loop away from the fat diagonal.

A geometric Dax isomorphism. We will now describe an explicit isomorphism

$$\mathsf{Dax} : \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}), M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \longrightarrow \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)].$$

We need to define explicit elements in $\pi_1(M)$, for which we pick a basepoint $b \in \partial M$, together with whiskers $\phi_i : \mathbb{I} \longrightarrow M$ with $\phi_i(0) = b$ and $\phi_i(1) = u_i(-1)$ for $u_i : \mathbb{D}^1 \hookrightarrow M$ a component of the chosen neat embedding $u : \mathbb{D}^{1,n} \hookrightarrow M$. In this case, we say the embedding u is *whiskered*. Furthermore, for each connected component of $\mathbb{D}^1_i$ of $\mathbb{D}^{1,n}$ and points $v \in \mathbb{D}^1_i$, we define whiskers $\varphi_i^v : \mathbb{I} \longrightarrow \mathbb{D}^1_i$ with $\varphi_i^v(0) = -1$ and $\varphi_i^v(1) = v$. Recall that we can assume maps

$$F : (\mathbb{I}^2, \mathbb{I} \times \{0\}, \partial\mathbb{I} \times \mathbb{I} \cup \mathbb{I} \times \{1\}) \longrightarrow (\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$$

representing an element $[F] \in \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$ to be perfect. Hence, the associated track $\widetilde{F} : \mathbb{I}^2 \times \mathbb{D}^{1,n} \longrightarrow \mathbb{I}^2 \times M$ has no triple point, and only finitely many isolated double points $p_1, \ldots, p_m$. As before, there are two different kinds of double points.

- Those, whose pre-image points $\{(\vec{t}_j, x_j), (\vec{t}_j, y_j)\} = \tilde{F}^{-1}(\{\vec{t}_j, p_j\})$ lie in the same connected component of $\mathbb{I}^2 \times \mathbb{D}^{1,n}$.
- Those, whose pre-image points $\{(\vec{t}_j, x_j), (\vec{t}_j, y_j)\} = \tilde{F}^{-1}(\{\vec{t}_j, p_j\})$ lie in different connected components of $\mathbb{I}^2 \times \mathbb{D}^{1,n}$.

To each such double point, we associate both an element $g_j \in \pi_1(M)$ and a sign $\varepsilon_j \in \{\pm 1\}$ in the following way. Let us consider the first case. By possibly re-parametrising $\mathbb{I}^2 \times \mathbb{D}^{1,n}$, we can assume that both x_j and y_j lie in a component $\mathbb{I} \times \{t_2\} \times \mathbb{D}^1$ with $x_j < y_j$. Scanning through the component is shown below in Figure 7.

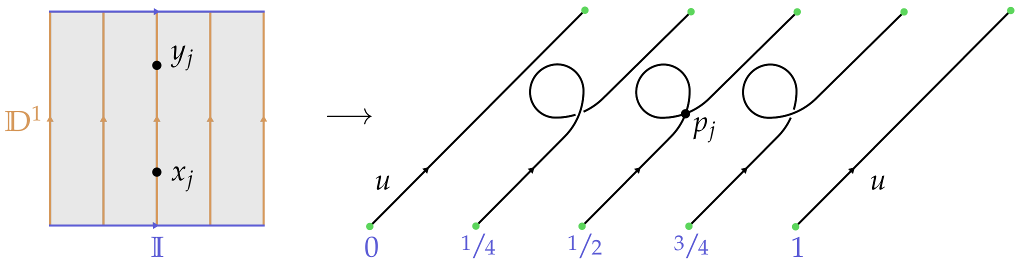


FIGURE 7. Scanning through $\mathbb{I} \times \{t_2\} \times \mathbb{D}^1$ which contains the pre-images x_j and y_j of the double point p_j.

The element $g_j \in \pi_1(M)$ is represented by the loop

$$\mathcal{O}_j := \phi_i \cdot F(\vec{t}_j)(\varphi_i^{x_j}) \cdot F(\vec{t}_j)(\varphi_i^{y_j})^{-1} \cdot \phi_i^{-1}$$

which is depicted on the left side in Figure 9. This should be compared to the definition of the loop in [KT23b, Theorem 4.14]. Let us now consider the second case, when the pre-image points lie in two different connected components, let us say $\mathbb{D}^1_{i_1}$ and $\mathbb{D}^1_{i_2}$ for $i_1 < i_2$. Scanning through the components is shown in Figure 8.

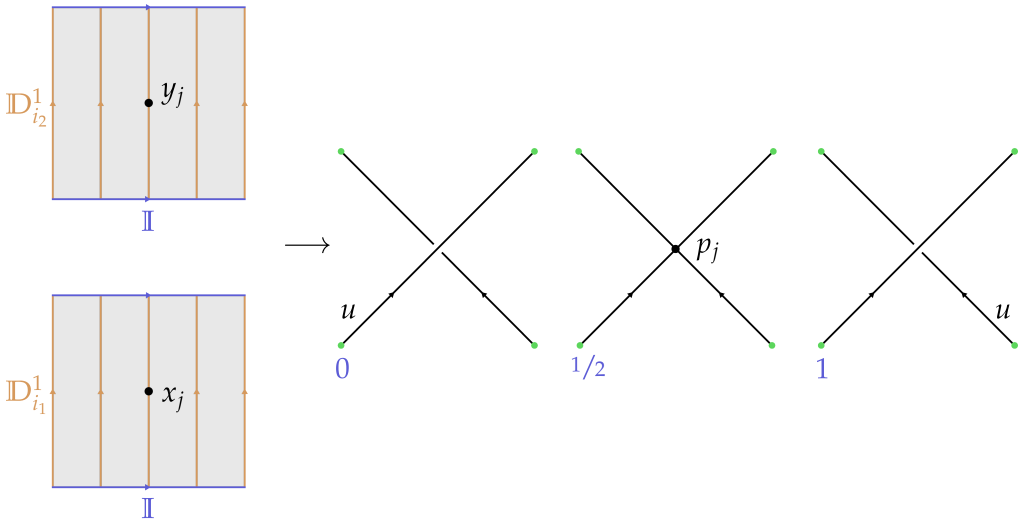


FIGURE 8. Scanning through $\mathbb{I} \times \{t_2\} \times \mathbb{D}^1_{i_1}$ and $\mathbb{I} \times \{t_2\} \times \mathbb{D}^1_{i_2}$ which contain the pre-images x_j and y_j of the double point p_j.

The associated element $g_j \in \pi_1(M)$ is represented by the loop

$$\mathcal{O}_j := \phi_{i_1} \cdot F(\vec{t}_j)(\varphi_{i_1}^{x_j}) \cdot F(\vec{t}_j)(\varphi_{i_2}^{y_j})^{-1} \cdot \phi_{i_2}^{-1}$$

and is depicted on the right side in Figure 9. Note that the definition is indeed very similar to the one in the first case. The difference is that we switch arcs as soon as we hit the double point p_j, and then travel back along the other arc. The formula holds for $i_1 \leq i_2$ and includes the previous case. We made the distinction to make the reader aware of the two different kinds of intersection. Note that the ordering of the path-components of $\mathbb{D}^{1,n}$ is crucial for the definition of the loops. Furthermore, in the first case, the ordering is given by the fact that the double points are isolated. This way, we can compare the $\mathbb{D}^1$-coordinate of the pre-image points x_j and y_j and choose the smaller valued one first (which was previously given by the assumption $x_j < y_j$).

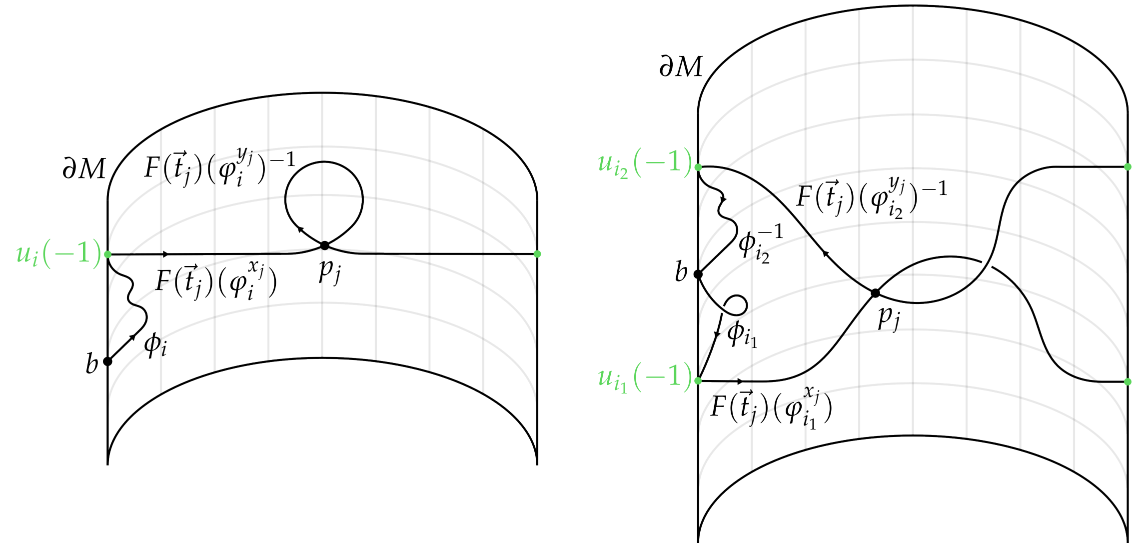


FIGURE 9. On the left side, the loop $\phi_i \cdot F(\vec{t}_j)(\varphi_i^{x_j}) \cdot F(\vec{t}_j)(\varphi_i^{y_j})^{-1} \cdot \phi_i^{-1}$ associated to the self-intersection of a single arc. On the right side, the loop $\phi_{i_1} \cdot F(\vec{t}_j)(\varphi_{i_1}^{x_j}) \cdot F(\vec{t}_j)(\varphi_{i_2}^{y_j})^{-1} \cdot \phi_{i_2}^{-1}$ associated to the intersection of two arcs. Both cases can be seen as "changing sheets".

In this case, the sign $\varepsilon_j \in \{\pm 1\}$ is obtained from comparing the orientation of the tangent space $T_{(\vec{t}_j, p_j)}(\mathbb{I}^2 \times M)$ and

$$d\widetilde{F}(T_{(\vec{t}_j, x_j)}(\mathbb{I}^2 \times \mathbb{D}^{1,n})) \oplus d\widetilde{F}(T_{(\vec{t}_j, y_j)}(\mathbb{I}^2 \times \mathbb{D}^{1,n})).$$

The last part of the data is coming from the set $\mathbb{T}_n$ which gives the information which connected components of $\mathbb{D}^{1,n}$ are intersecting each other, and hence the type of intersection as just discussed. An element $(i_1, i_2) \in \mathbb{T}_n$ denotes the intersection of the component i_1 with the component i_2. Note that this includes self-intersection of a single arc by considering $i_1 = i_2$. For the element g_j associated to the intersection of arcs i_{j_1} and i_{j_2}, we write $T_j := (i_{j_1}, i_{j_2})$ and $g_{j,T_j} = (T_j, g_j) \in \mathbb{T}_n \times \pi_1(M)$. We then define $\mathsf{Dax}([F]) := \sum_{j=1}^{m} \varepsilon_j g_{j,T_j}$.

Theorem 26. *Let M be a compact, oriented, connected 4-manifold with non-empty boundary ∂M and a basepoint $b \in \partial M$. For a choice of a whiskered neat embedding $u : \mathbb{D}^{1,n} \hookrightarrow M$, the map*

$$\mathsf{Dax} : \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \longrightarrow \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)]$$

defined by $\mathsf{Dax}([F]) := \sum_{j=1}^{m} \varepsilon_j g_{j,T_j}$ as above is an isomorphism.

Note that additivity of Dax comes from gluing together two vertical faces of $\mathbb{I}^2$, which is composition in the second relative homotopy group. Under Dax, this yields added intersection counts.

Lemma 27. *The projection*

$$E_u^{\Upsilon} \xrightarrow{\mathsf{pr}_{(\mathbb{D}^{1,n})^2}} \mathsf{Conf}_2(\mathbb{D}^{1,n})$$

is a fibration sequence with fibre space homotopy equivalent to ΩM.

Proof. Let us start by considering an element $[v_1, v_2] \in \mathsf{Conf}_2(\mathbb{D}^{1,n})$. The fibre $\mathsf{pr}^{-1}_{(\mathbb{D}^{1,n})^2}([v_1, v_2])$ contains paths $\gamma \in \mathsf{Map}([-1,1], M)$ such that $\gamma(-1) = u(v_1)$ and $\gamma(1) = u(v_2)$, meaning paths in M that start at $u(v_1)$ and end in $u(v_2)$. There is a homotopy equivalence $\Omega M \simeq \mathsf{pr}^{-1}_{(\mathbb{D}^{1,n})^2}([v_1, v_2])$ by the following two homotopy inverses. Let us assume that v_1 lies in the component $\mathbb{D}^1_{i_1}$ and v_2 in the component $\mathbb{D}^1_{i_2}$ of $\mathbb{D}^{1,n}$. Note that we do not exclude the case of $i_1 = i_2$.

- $f : \mathsf{pr}^{-1}_{(\mathbb{D}^{1,n})^2}([v_1, v_2]) \longrightarrow \Omega M$ sending such a path $\gamma \in \mathsf{Map}([-1,1], M)$ starting at $u(v_1)$ and ending in $u(v_2)$ to the based loop
$$f(\gamma) := \phi_{i_1} \cdot u(\varphi_{i_1}^{v_1}) \cdot \gamma \cdot u((\varphi_{i_2}^{v_2})^{-1}) \cdot \phi_{i_2}^{-1}.$$
- $g : \Omega M \longrightarrow \mathsf{pr}^{-1}_{(\mathbb{D}^{1,n})^2}([v_1, v_2])$ sending a based loop $\rho \in \Omega M$ to the path
$$g(\rho) := u((\varphi_{i_1}^{v_1})^{-1}) \cdot \phi_{i_1}^{-1} \cdot \rho \cdot \phi_{i_2} \cdot u(\varphi_{i_2}^{v_2})$$
starting at $u(v_1)$ and ending in $u(v_2)$.

These maps are clearly homotopy inverses, hence $\Omega M \simeq \mathsf{pr}^{-1}_{(\mathbb{D}^{1,n})^2}([v_1, v_2])$ and we indeed have a fibration sequence

$$\Omega M \longrightarrow E_u^{\Upsilon} \xrightarrow{\mathsf{pr}_{(\mathbb{D}^{1,n})^2}} \mathsf{Conf}_2(\mathbb{D}^{1,n}).$$

Note that this can also be obtained by restricting the fibration sequence mentioned in the proof of Lemma 20 to the subspace $\dot{W}$. □

Remark 28. This fibration sequence gives an alternative and, admittedly, simpler proof of Theorem 22.

Proof of Theorem 26. We start with a formal discussion, unravelling the given data. The main argument then boils down to showing that the loops

$$\mathcal{O}_j := \phi_{i_1} \cdot F(\vec{t}_j)(\varphi_{i_1}^{x_j}) \cdot F(\vec{t}_j)(\varphi_{i_2}^{y_j})^{-1} \cdot \phi_{i_2}^{-1}$$

defined above are based homotopic to the loops appearing in the definition of the Dax isomorphism, see Figure 6. A cobordism class of $b_{\mathrm{Dax}} \longrightarrow E_u^{\Upsilon}$ is given by the sum of *signed* connected components of E_u^{Υ} containing the image $\mathrm{im}(b_{\mathrm{Dax}})$. The sign $\varepsilon_{\mathrm{Dax}} \pm 1$ comes from the bundle isomorphism $\mathscr{B}_{\mathrm{Dax}} : b^*_{\mathrm{Dax}}(\vartheta_u|_{E_u^{\Upsilon}}) \longrightarrow \nu_{\Delta_{\mathrm{Dax}}}$. It is positive if $\mathscr{B}_{\mathrm{Dax}}$ preserves the orientation, negative otherwise. We can similarly pass to the setting of the space $\widetilde{E}^{\Upsilon}_u$ before considering the $\mathbb{Z}/2$-action given by the involution. This has the advantage of being able to use the fibration sequence of Lemma 27 while keeping track of *direction*. Let us consider a map

$$F : (\mathbb{I}^2, \mathbb{I} \times \{0\}, \partial\mathbb{I} \times \mathbb{I} \cup \mathbb{I} \times \{1\}) \longrightarrow (\mathrm{Imm}_\partial(\mathbb{D}^{1,n}), M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$$

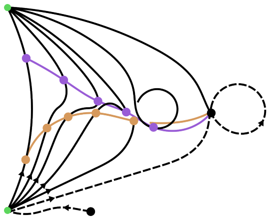
representing an element $[F] \in \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}), M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$ and a tuple $(\vec{t}, x, y) \in \widetilde{\Delta}_{\mathrm{Dax}}$, with associated double point $F(\vec{t})(x) = p_j = F(\vec{t})(y)$. Under the identification of the fibre space of the fibration

$$E^{\Upsilon}_u \xrightarrow{\mathsf{pr}_{(\mathbb{D}^{1,n})^2}} \mathsf{Conf}_2(\mathbb{D}^{1,n})$$

as discussed in Lemma 27, the image of $\widetilde{b}_{\mathrm{Dax}}((\vec{t}, x, y))$ given as the path

$$\gamma := F(\mathscr{H}_{\vec{t}})(x) \cdot F(\mathscr{H}_{\vec{t}})(y)^{-1}$$

from $u(x)$ to $u(y)$ corresponds to the class $[f(\gamma) = \phi_{i_1} \cdot u(\varphi^x_{i_1}) \cdot \gamma \cdot u((\varphi^y_{i_2})^{-1}) \cdot \phi^{-1}_{i_2}]$. Here f is the homotopy equivalence defined in the proof of Lemma 27, and we assume that x lies in the component $\mathbb{D}^1_{i_1}$ and y in $\mathbb{D}^1_{i_2}$ of $\mathbb{D}^{1,n}$. We need to show that the loop $f(\gamma)$ is based homotopic to the loop $\mathcal{O}_j$ defined above. Then, they represent the same element in $\pi_1(M)$ and we are only left with checking that the sign agrees.



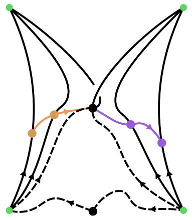


FIGURE 10. The arcs coming from the original Dax isomorphism in colour, the new loops defined in the geometric Dax isomorphism dashed. The left-hand side shows the case of $i_1 = i_2$, whereas the right-hand side shows the case $i_1 \neq i_2$.

To do so, we refer to Figure 6 and Figure 10, and remark that the defined 1-parameter family of arcs is foliating a disk. The disk can be degenerate, for example in the case of the constant map. Given a double point p_j, by definition, the loop $\mathcal{O}_j$ lies on that disk. Indeed, a tuple $(\vec{t}, x, y) \in \tilde{\Delta}_{\mathrm{Dax}}$ associated to the double point p_j yields $F(\vec{t})(x) = p_j = F(\vec{t})(y)$, and $\mathcal{O}_j$ lies on the immersed arc $F(\vec{t})$ after choosing whiskers ϕ_{i_1} and ϕ_{i_2}. The case of $i_1 \neq i_2$ is depicted on the right-hand side of Figure 10, the dashed loop being $\mathcal{O}_j$. The loop $f(\gamma)$ is given by following the dashed whisker to the left, then the utmost left arrow, which is in the position of u_{i_1}, then along the arc γ following the points, and the analogous way back on the right-hand side. Ignoring whiskers, both arcs lie on the same foliated disk, and hence one can homotope one loop into another, fixing the point p_j of changing sheets. The case of $i_1 = i_2$ is depicted on the left-hand side of Figure 10 and has been discussed in [KT23b, Theorem 4.14]. Note that in the figure, one of the arcs defined in the original Dax isomorphism wraps around the formed loop. A chosen homotopy fixes this part and drags everything else to the dotted line along the foliated disk. This shows that the loops $f(\gamma)$ and $\mathcal{O}_j$ are based homotopic and represent the same element in the fundamental group $\pi_1(M)$.

It is left to check that the associated sign agrees. In the definition of the geometric Dax isomorphism, the sign $\varepsilon_j = \pm 1$ associated to the double point p_j comes from comparing the orientation of the tangent space $T_{(\vec{t}_j, p_j)}(\mathbb{I}^2 \times M)$ to the one of

$$d\widetilde{F}(T_{(\vec{t}_j,x)}(\mathbb{I}^2 \times \mathbb{D}^{1,n})) \oplus d\widetilde{F}(T_{(\vec{t}_j,y)}(\mathbb{I}^2 \times \mathbb{D}^{1,n})).$$

Recall the quotient map $q : \widetilde{\Delta}_{\mathrm{Dax}} \longrightarrow \Delta_{\mathrm{Dax}}$. Passing to the covering space, as mentioned above, the sign $\varepsilon_{\mathrm{Dax}} = \pm 1$ coming from the original Dax isomorphism depends on whether the bundle isomorphism $\widetilde{\mathscr{B}}_{\mathrm{Dax}} : \widetilde{b}^*_{\mathrm{Dax}}(\widetilde{\vartheta}_u) \longrightarrow q^* \nu_{\Delta_{\mathrm{Dax}}}$ preserves orientation. Furthermore, recall the previously discussed isomorphisms

$$\begin{aligned} \nu_{\widetilde{\Delta}_{\mathrm{Dax}} \subseteq (\mathbb{I}^2 \times \mathbb{D}^{1,n})^2} &\cong_s (\widetilde{F}^2_h)^*(\nu_{\Delta_{\mathbb{I}^2 \times M} \subseteq (\mathbb{I}^2 \times M)^2}) \cong_s (\widetilde{F}^2_h)^*(\mathsf{pr}_1^*(T(\mathbb{I}^2 \times M)) \\ &\cong (\mathsf{pr}_1 \circ \widetilde{F}^2_h)^*(TM) \cong \widetilde{b}^*_{\mathrm{Dax}}(\mathsf{pr}^*_M(TM)). \end{aligned}$$

Unravelling this isomorphism together with the splitting

$$\nu_{\widetilde{\Delta}_{\mathrm{Dax}}} \cong \nu_{(\mathbb{I}^2 \times \mathbb{D}^{1,n})^2}|_{\widetilde{\Delta}_{\mathrm{Dax}}} \oplus \nu_{\widetilde{\Delta}_{\mathrm{Dax}} \subseteq (\mathbb{I}^2 \times \mathbb{D}^{1,n})^2} \cong_s \nu^2_{\mathbb{D}^{1,n}}|_{\widetilde{\Delta}_{\mathrm{Dax}}} \oplus (\widetilde{F}^2_h)^*(\nu_{\Delta_{\mathbb{I}^2 \times M} \subseteq (\mathbb{I}^2 \times M)^2}),$$

the sign $\varepsilon_{\mathrm{Dax}} = \pm 1$ associated to the tuple $(\vec{t}, x, y)$ is precisely $+1$ if and only if $d(\widetilde{F}^2_h)$ is orientation preserving at $(\vec{t}, x, y)$ which is equivalent to the orientation of $\nu_{\Delta_{\mathbb{I}^2 \times M} \subseteq (\mathbb{I}^2 \times M)^2}$ agreeing with the orientation of

$$\begin{aligned} d\widetilde{F}^2|_{(\vec{t},x,y)}(T(\mathbb{I}^2 \times \mathbb{D}^{1,n})^2) &= (d\widetilde{F}|_{(\vec{t},x)}(T(\mathbb{I}^2 \times \mathbb{D}^{1,n})), d\widetilde{F}|_{(\vec{t},y)}(T(\mathbb{I}^2 \times \mathbb{D}^{1,n}))) \\ &\cong d\widetilde{F}(T_{(\vec{t}_j,x)}(\mathbb{I}^2 \times \mathbb{D}^{1,n})) \oplus d\widetilde{F}(T_{(\vec{t}_j,y)}(\mathbb{I}^2 \times \mathbb{D}^{1,n})) \end{aligned}$$

after identifying the tangent bundle with the normal bundle of the diagonal in the product. This is precisely the definition of the sign $\varepsilon_j = \pm 1$ in the definition of the geometric Dax isomorphism. ☐

The realisation map. Let M be a compact, oriented, connected 4-manifold with non-empty boundary ∂M and a basepoint $b \in \partial M$. Let $u : \mathbb{D}^{1,n} \hookrightarrow M$ be a whiskered neat embedding. An explicit inverse to the isomorphism Dax is given by the following *realisation map*

$$\mathfrak{r} : \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)] \longrightarrow \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u).$$

Given $((i_1, i_2), g) \in \mathbb{T}_n \times \pi_1(M)$, the idea is to describe an explicit map

$$\mathfrak{r}(((i_1, i_2), g)) : (\mathbb{I}^2, \mathbb{I} \times \{0\}, \partial\mathbb{I} \times \mathbb{I} \cup \mathbb{I} \times \{1\}) \longrightarrow (\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$$

defining an element in $\pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$ by taking its homotopy class and use linear extension. Let us fix $0 < \varepsilon \ll 1$ and partition $\mathbb{I}^2$ in the way depicted in Figure 11. We trace through a generic slice $\mathbb{I} \times \{t\}$, which yields a path through immersions (embeddings in the case of $t = 0$) starting and ending at u. For fixed $(i_1, i_2) \in \mathbb{T}_n$, we only need to describe a family of immersions on the corresponding arcs, the other arcs simply stay in their basepoint configuration. The reader should keep that in mind.

For $i_1 \leq i_2$, we describe a slice of part ① ($[0, \varepsilon] \times \{t\}$ for $0 \leq t \leq 1 - \varepsilon$) by moving the i_1-th component of $\mathbb{D}^{1,n}$ through M, while all other components are fixed to stay in the position of their respective component of the basepoint u. Take a neighbourhood near $u_{i_1}(-1 + \varepsilon)$ and push it along the inverse-whisker $\phi_{i_1}^{-1}$, around a loop representing g, and back along the whisker ϕ_{i_2}, stopping before any intersection can occur. We say the disk is in *pre-loop* position. To make sure that the arc is embedded at any stage, we pick tubular neighbourhoods of all paths to ensure enough space. A slice through part ③ is the exact inverse path, ending at u. To describe part ②, we consider a meridian $\mu_{\mathbf{x}}(\mathbb{S}^2)$ around the point $\mathbf{x} := u_{i_2}(-1 + 2\varepsilon)$. This means the meridian 2-sphere bounds a meridian 3-ball $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^3)$ intersecting u_{i_2} exactly once in $\mathbf{x}$. For $(\varepsilon, 1 - \varepsilon) \times \{0\}$, we foliate the meridian $\mu_{\mathbf{x}}(\mathbb{S}^2)$ by a 1-family of 1-disks based at two fixed points on the piece we previously pushed along $\phi_{i_1}^{-1} \cdot g \cdot \phi_{i_2}$. In [KT23b, Theorem 4.21], this is fittingly called “swinging a lasso”. For $(\varepsilon, 1 - \varepsilon) \times \{t\}$ with $0 < t \leq 1 - \varepsilon$, we analogously foliate spheres with decreasing diameter, yet still based at two fixed points. This is done in such a way, that the map is constant for $t = 1 - \varepsilon$. In this sense, we foliate $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^3)$ by a 1-family of 2-spheres, each sphere being foliated by a 1-family of 1-disks. Note that there exists exactly one point for which the arc intersects u_{i_2} once in $\mathbf{x}$. In part ④, we continuously undo the pushing along $\phi_{i_1}^{-1} \cdot g \cdot \phi_{i_2}$ such that the map is constantly u on $\mathbb{I} \times \{1\}$.

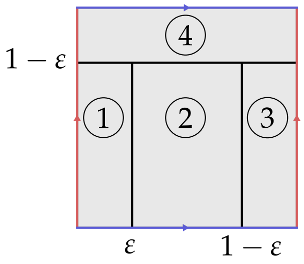


FIGURE 11. A partition of $\mathbb{I}^2$.

This defines an element $\mathfrak{r}((i_1, i_2), g)$. For $\mathfrak{r}(-((i_1, i_2), g))$, we connect into the meridian 3-ball $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^3)$ from the other side by pushing the arc around u_{i_2} first. We then extend it linearly by choosing disjointed supports and meridian balls. In the above description, this amounts to varying ε accordingly.

We furthermore define the point-wise restriction of $\mathfrak{r}$ to $\mathbb{I} \times \{0\}$ as $\partial\mathfrak{r}$. The image is a loop in $\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$ based at u, giving an element in $\pi_1(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$.

Theorem 29. *The realisation map*

$$\mathfrak{r} : \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)] \longrightarrow \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u).$$

as described above is the inverse of Dax *as in Theorem 26.*

Proof. We show that the realisation map is a right inverse to Dax, hence $(\mathrm{Dax} \circ \mathfrak{r})((i_1, i_2), g) = ((i_1, i_2), g) \in \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)]$. Since the latter is an isomorphism by Theorem 26, hence we then

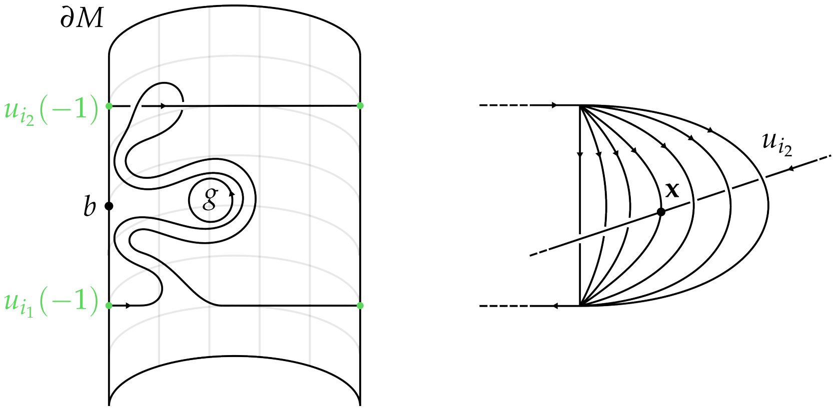


FIGURE 12. On the left-hand side, $\mathfrak{r}((i_1, i_2), g)$ evaluated at $(1/2, 0)$. Following the whisker ϕ_{i_1} to the basepoint b, around the π_1-element g, then following $\phi_{i_2}^{-1}$, and half-way around the meridian sphere $\mu_{\mathbf{x}}(\mathbb{S}^2)$. On the right-hand side, the part around u_{i_2} of $\mathfrak{r}((i_1, i_2), g)$ evaluated on $(1/2, t)$ for $t \in [0, 1-\varepsilon]$. The case $i_1 = i_2$ is also depicted in [KT23b, Figure 4.22].

can conclude that $\mathfrak{r}$ is indeed an inverse to Dax. Let us choose $((i_1, i_2), g) \in \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)]$, and consider $\mathfrak{r}((i_1, i_2), g)$ as defined above. Note that the meridian 3-ball $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^3)$ intersects u_{i_2} exactly once and transversely in $\mathbf{x}$. Thus, the only double point happens when the foliation goes through the centre point of $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^3)$, namely $\mathbf{x}$. This state is depicted in Figure 13.

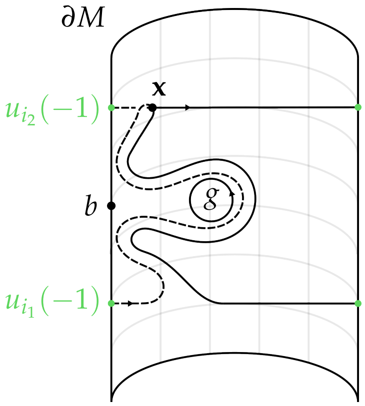


FIGURE 13. Ignoring whiskers, the dotted line represents the element obtained after applying Dax to $\mathfrak{r}((i_1, i_2), g)$. Note that the intersection at $\mathbf{x}$ is *transverse*.

One can easily read off the element obtained after applying Dax. It is given by the homotopy class of $\phi_{i_1} \cdot \phi_{i_1}^{-1} \cdot g \cdot \phi_{i_2} \cdot \phi_{i_2}^{-1}$, which is the same as g itself. Thus, the π_1-element is the correct one, and we only need to check the sign ± 1 as the intersecting arcs corresponding to $(i_1, i_2) \in \mathbb{T}_n$ obviously do not change. Recall, the sign $\varepsilon_{\mathbf{x}} \in \{\pm 1\}$ is obtained from comparing the orientation of the tangent space $T_{(\vec{t}_{\mathbf{x}},\mathbf{x})}(\mathbb{I}^2 \times M)$ and

$$d\mathfrak{r}(\widetilde{(i_1,i_2)},g)(T_{(\vec{t}_{\mathbf{x}},x)}(\mathbb{I}^2 \times \mathbb{D}^{1,n})) \oplus d\mathfrak{r}(\widetilde{(i_1,i_2)},g)(T_{(\vec{t}_{\mathbf{x}},y)}(\mathbb{I}^2 \times \mathbb{D}^{1,n}))$$

with x and y being the two pre-images of the double point $\mathbf{x}$ of the track of $\mathfrak{r}((i_1,i_2),g)$, given as the map

$$\mathfrak{r}(\widetilde{(i_1,i_2)},g) : \underbrace{\mathbb{I}^2 \times \mathbb{D}^{1,n}}_{\text{dimension 3}} \longrightarrow \underbrace{\mathbb{I}^2 \times M}_{\text{dimension 6}} \quad \text{sending } (\vec{t},v) \text{ to } (\vec{t},\mathfrak{r}((i_1,i_2),g)(\vec{t})(v)).$$

We consider an open neighbourhood cube around the double point $\mathbf{x} \in M$, modelled as $\mathbb{R}^4$. We can choose coordinates such that the transverse intersection happens in the 2-plane $\mathbb{R}^2 \times \{0\} \times \{0\} \subseteq \mathbb{R}^4$, the two arcs modelled as the i_2-sheet $\mathbb{R} \times \{0\} \times \{0\} \times \{0\}$ and the i_1-sheet $\{0\} \times \mathbb{R} \times \{0\} \times \{0\}$. In this chart, the derivate of the i_1-sheet of $\mathfrak{r}((i_1,i_2),g)$ applied to $\mathbb{I} \times \{0\} \subseteq \mathbb{I}^2$ yields the positive basis of $\mathbb{I} \times \{0\} \subseteq \mathbb{I}^2$ and the positive basis of $\{0\} \times \{0\} \times \{0\} \times \mathbb{R} \subseteq \mathbb{R}^4$. Applied to $\{0\} \times \mathbb{I} \subseteq \mathbb{I}^2$, we obtain the sum of the positive basis of $\{0\} \times \mathbb{I} \subseteq \mathbb{I}^2$ and the positive basis of $\{0\} \times \{0\} \times \mathbb{R} \times \{0\} \subseteq \mathbb{R}^4$. The derivative of the i_1-sheet applied to $\mathbb{D}^1_{i_2}$ yields the positive basis $\{0\} \times \mathbb{R} \times \{0\} \times \{0\}$. The case of the i_2-sheet is simpler, as u_{i_2} is constant. We simply obtain the positive basis of $\mathbb{I} \times \{0\} \subseteq \mathbb{I}^2$ and the positive basis of $\mathbb{R} \times \{0\} \times \{0\} \times \{0\}$. Using transpositions, we can compare the basis of the model to the canonical basis of $\mathbb{I}^2 \times \mathbb{R}^4 \subseteq \mathbb{I}^2 \times M$ and obtain a positive sign $\varepsilon_{\mathbf{x}} = 1$. As a note, one can view this as a choice of either left-hand or right-hand rule, comparing the bases of the tangent spaces. If we consider $\mathfrak{r}(-((i_1,i_2),g))$, we previously said that we connect from "the other side". This exactly changes the left-hand rule to the right-hand rule and vice versa, hence the sign after applying Dax is precisely the other one, in this case -1. We have shown that $\mathfrak{r}$ is right-inverse, and since Dax is known to be an isomorphism, it is the unique inverse. ☐

The fact that we have a realisation map as an inverse gives us a way to translate the algebraic point of view back into the geometric one. This is going to be useful when identifying families of arcs having certain algebraic properties.

IV. CALCULATING HOMOTOPY GROUPS

In this section, we further study the long exact sequence of homotopy groups associated to the pair $(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M))$ of neat immersions and embeddings of multi-arcs into a 4-manifold M. In particular, we begin with a discussion on the homotopy type of the immersion space $\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M)$, after which we study the map

$$\delta_{\mathrm{Imm}} : \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), u) \longrightarrow \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u).$$

We further compare the homotopy types of the embedding $\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$ and the mapping space $\mathrm{Map}_\partial(\mathbb{D}^{1,n}, M)$ yielding the following diagram which should be compared to [KT23a, (3.7)].

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, "c"] \arrow[d, tail, "\iota_*"] & \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)] \arrow[d, shift left, tail, two heads, "\mathfrak{t}"] & & \\
\pi_2(\mathrm{Imm}_\partial, u) \arrow[r, "\delta_{\mathrm{Imm}}"] \arrow[d, two heads, "p_*"] & \pi_2(\mathrm{Imm}_\partial, \mathrm{Emb}_\partial, u) \arrow[u, shift left, tail, two heads, "\mathsf{Dax}"] \arrow[r, "\partial"] \arrow[d] & \pi_1(\mathrm{Emb}_\partial, u) \arrow[r, two heads] \arrow[d, equal] & \pi_1(\mathrm{Imm}_\partial, u) \arrow[d, tail, two heads, "p_*"] \\
\pi_2(\mathrm{Map}_\partial, u) \arrow[r, "\delta_{\mathrm{Map}}"] \arrow[d, tail, two heads, "p_u"] & \pi_2(\mathrm{Map}_\partial, \mathrm{Emb}_\partial, u) \arrow[r, "\partial"] \arrow[d, shift left, tail, two heads, "\mathsf{Dax}"] & \pi_1(\mathrm{Emb}_\partial, u) \arrow[r, two heads] & \pi_1(\mathrm{Map}_\partial, u) \arrow[d, tail, two heads, "p_u"] \\
\prod_{i=1}^n \pi_3(M) \arrow[r, "\mathsf{dax}_u"] & \mathbb{Z}[\pi_1(M)^{\dagger,n}] \arrow[u, shift left, tail, two heads, "\mathfrak{t}"] & & \prod_{i=1}^n \pi_2(M)
\end{tikzcd}$$

Note that the cokernel of the map δ_{Imm} is precisely the kernel of the surjection $\pi_1(\mathrm{Emb}_\partial, u) \longrightarrow \pi_1(\mathrm{Imm}_\partial, u)$. We will build and analyse this diagram, which will lead to a proof of Theorem B.

IV.1. ON HOMOTOPY GROUPS OF IMMERSION SPACES

Lemma 30 ([KT23b, Lemma 4.26])**.** *For a based space Y let $f : \mathbb{D}^k \longrightarrow Y$ be a based map, mapping the basepoint $e_1 \in \partial\mathbb{D}^k$ to the basepoint $f(e_1) = e_Y$ of Y. Then there are inverse homotopy equivalences*

$$-f \cup_\partial \bullet : \mathrm{Map}_\partial(\mathbb{D}^k, Y; f) \rightleftarrows \mathrm{Map}_*(\mathbb{S}^k, Y) \cong \Omega^k Y : f \vee \bullet$$

based for f and $-f \cup_\partial f$. The map $-f \cup_\partial \bullet$ sends an element $d \in \mathrm{Map}_\partial(\mathbb{D}^k, Y; f)$ to $-f \cup_\partial d : \mathbb{S}^k \longrightarrow Y$, gluing the two disks together on the common boundary, resulting in a sphere. The map $f \vee \bullet$ sends an element $s \in \mathrm{Map}_*(\mathbb{S}^k, Y)$ to the pinch map

$$f \vee s : \mathbb{D}^k \xrightarrow{\text{pinch}} \mathbb{D}^k \vee \mathbb{S}^k \xrightarrow{f \vee s} Y.$$

Remark 31. The same holds true if we extend the boundary condition on $\partial\mathbb{D}^k$ to one the collar $\partial^\varepsilon\mathbb{D}^k$. In that case, there is a homotopy equivalence $\mathrm{Map}_{\partial^\varepsilon}(\mathbb{D}^k, Y) \simeq \Omega^k Y$.

We begin with a brief discussion on the homotopy groups of the immersion space $\mathrm{Imm}_{\partial}(\mathbb{D}^{1,n},M)$ of immersions of multi-arcs into a compact, oriented, connected 4-manifold M. Hirsch-Smale theory, see [Sma58], [Hir59] and [Sma59], yields a weak homotopy equivalence

$$\mathcal{D} : \mathrm{Imm}_{\partial}(\mathbb{D}^{1,n},M) \longrightarrow \mathrm{Imm}^{\mathsf{f}}_{\partial}(\mathbb{D}^{1,n},M)$$

given by $f \mapsto (f, df)$. Since $\mathbb{D}^{1,n}$ is compact, both spaces are of the homotopy type of CW-complexes, and therefore it is a homotopy equivalence by Whitehead's theorem. This follows from work of Milnor in [Mil59]. The space $\mathrm{Imm}^{\mathsf{f}}_{\partial}(\mathbb{D}^{1,n},M)$ is the space of *formal immersions*, meaning pairs (f, Tf) with $f : \mathbb{D}^{1,n} \longrightarrow M$ a continuous map and $Tf : T\mathbb{D}^{1,n} \longrightarrow f^{*}TM$ a monomorphism of tangent bundles. One can similarly require f to be smooth as the smooth mapping space is homotopy equivalent to the continuous one. This is the first example of the h-principle. Note that $T\mathbb{D}^{1,n} \cong \mathbb{D}^{1,n} \times \mathbb{R}$ is the trivial bundle. The data of an element $(f, Tf) \in \mathsf{f}\,\mathrm{Imm}^{\mathsf{f}}_{\partial}(\mathbb{D}^{1,n},M)$ is given by a commuting diagram

$$\begin{tikzcd}
\mathbb{D}^{1,n} \times \mathbb{R} \arrow[r, tail, "Tf"] \arrow[d] & TM \arrow[d] \\
\mathbb{D}^{1,n} \arrow[r, "f"] & M
\end{tikzcd}$$

and since the map Tf is required to be a monomorphism, there is a homotopy equivalence $\mathrm{Imm}^{\mathsf{f}}_{\partial}(\mathbb{D}^{1,n},M) \simeq \mathrm{Map}_{\partial}(\mathbb{D}^{1,n}, TM \setminus s_0(M))$ with $s_0 : M \longrightarrow TM$ the zero-section. Since the space $T \setminus s_0(M)$ is homotopy equivalent to the sphere bundle $\mathbb{S}TM$, we get another homotopy equivalence $\mathrm{Imm}^{\mathsf{f}}_{\partial}(\mathbb{D}^{1,n},M) \simeq \mathrm{Map}_{\partial}(\mathbb{D}^{1,n}, \mathbb{S}TM)$. Taking coproducts to products and applying Lemma 30, we obtain

$$\mathrm{Imm}_{\partial}(\mathbb{D}^{1,n},M) \simeq \mathrm{Map}_{\partial}(\mathbb{D}^{1,n}, \mathbb{S}TM) \cong \textstyle\prod_{i=1}^{n} \mathrm{Map}_{\partial_i}(\mathbb{D}^{1}, \mathbb{S}TM) \cong \textstyle\prod_{i=1}^{n} \Omega_{u_i(-1)}\mathbb{S}TM.$$

Observe that each mapping space $\mathrm{Map}_{\partial_i}(\mathbb{D}^{1}, \mathbb{S}TM)$ is itself homotopy equivalent to the immersion space $\mathrm{Imm}_{\partial_i}(\mathbb{D}^{1}, M)$ by applying Hirsch-Smale, hence we obtain the homotopy equivalence $\mathrm{Imm}_{\partial}(\mathbb{D}^{1,n},M) \simeq \prod_{i=1}^{n} \mathrm{Imm}_{\partial_i}(\mathbb{D}^{1}, M)$. Immersion spaces do not increase in complexity when considering multiple components, in contrast to embedding spaces. That is one reason why the approach of comparing the homotopy type of the embedding space to the homotopy type of the immersion space works well in our case of multi-disks.

Throughout this discussion generalised to k-multi-disks, we secretly identified $V_1(M)$, the (orthonormal) 1-frame bundle of the tangent bundle TM of M, with the sphere bundle $\mathbb{S}TM$. For a more general discussion, we refer to the following corollary which summarises the previous discussion, combining the Hirsch-Smale map with Lemma 30.

Lemma 32 ([KT23b, Corollary 4.27])**.** *There is a homotopy equivalence*

$$\mathcal{D}_u(\bullet) := -\mathcal{D}(u) \cup_{\partial} \mathcal{D}(\bullet) : \mathrm{Imm}_{\partial}(\mathbb{D}^{k}, M) \longrightarrow \Omega^{k} V_k(M)$$

with $V_k(M)$ being the total space of the orthonormal k-frame bundle of the tangent bundle of M.

With the above discussion of taking coproducts to products, this immediately yields the following corollary.

Corollary 33. *There is a homotopy equivalence*

$$\mathcal{D}_u(\bullet) := -\mathcal{D}(u) \cup_\partial \mathcal{D}(\bullet) : \mathrm{Imm}_\partial(\mathbb{D}^{k,n}, M) \longrightarrow \textstyle\prod_{i=1}^n \Omega^k V_k(M).$$

Remark 34. Note that applying Lemma 30 component-wise yields different basepoints for the loop space. In our case, we choose $u_i(-1)$. We suppress this in the upcoming notation, the reader should be aware. Products of homotopy groups obtained from this equivalence have accordingly changing basepoint. This can be uniformised by choosing whiskers, but it is going to be helpful to keep the information of each individual basepoint.

The fibres of the bundle map $\mathbb{S}TM \longrightarrow M$ are given by the sphere in 4-dimensional space, $\mathbb{S}^3$, yielding the fibre bundle $\mathbb{S}^3 \longrightarrow \mathbb{S}TM \longrightarrow M$. This discussion allows for the computation of the following homotopy groups.

Proposition 35. *Let M be compact, oriented and connected 4-manifold. For any chosen basepoint $u \in \mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M)$, we can compute the following homotopy groups.*

- $\pi_1(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), u) \cong \prod_{i=1}^n \pi_2(M)$.
- $\pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), u)$ *can be expressed a group extension of $\mathbb{Z}^n$ by $\prod_{i=1}^n \pi_3(M)$. This means that there is a short exact sequence*

  $$\mathbb{Z}^n \rightarrowtail \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), u) \twoheadrightarrow \textstyle\prod_{i=1}^n \pi_3(M)$$

  of abelian groups.

Proof. We apply the above discussion on results due to Hirsch and Smale. For the first case, notice that we have isomorphisms $\pi_1(\Omega\mathbb{S}TM) \cong \pi_2(\mathbb{S}TM) \cong \pi_2(M)$, where the latter one is obtained from the long exact sequence of homotopy groups associated to the fibre bundle $\mathbb{S}^3 \xrightarrow{\iota} \mathbb{S}TM \longrightarrow M$. For the second result, we consider the product fibre bundle

$$\textstyle\prod_{i=1}^n \mathbb{S}^3 \xrightarrow{\iota} \prod_{i=1}^n \mathbb{S}TM \longrightarrow \prod_{i=1}^n M$$

which yields the desired short exact sequence. Indeed, the map $\iota_* : \pi_3(\mathbb{S}^3) \longrightarrow \pi_3(\mathbb{S}TM)$ is clearly injective as every fibre bundle admits a section (the zero-section), hence we obtain an injection $\mathbb{Z}^n \rightarrowtail \pi_3(\prod_{i=1}^n \mathbb{S}TM) \cong \pi_2(\prod_{i=1}^n \Omega\mathbb{S}TM) \cong \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), u)$ by taking the product. □

IV.2. ANALYSING THE COMPOSITION MAP

So far, we collected the data in the following diagram

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, dashed, "c"] \arrow[d, tail, "\iota_*"] & \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)] \arrow[d, shift right, tail, two heads, "\mathfrak{r}"'] & & \\
\pi_2(\mathrm{Imm}_\partial, u) \arrow[r, "\delta_{\mathrm{Imm}}"] \arrow[d, two heads] & \pi_2(\mathrm{Imm}_\partial, \mathrm{Emb}_\partial, u) \arrow[r, "\partial"] \arrow[u, shift right, two heads, "\mathsf{Dax}"'] & \pi_1(\mathrm{Emb}_\partial, u) \arrow[r, two heads] & \pi_1(\mathrm{Imm}_\partial, u) \arrow[d, tail, two heads, "\cong"] \\
\prod_{i=1}^n \pi_3(M) & & & \prod_{i=1}^n \pi_2(M)
\end{tikzcd}$$

and we will analyse the map $c : \mathbb{Z}^n \longrightarrow \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)]$ given by the composition $\mathsf{Dax} \circ \delta_{\mathrm{Imm}} \circ \iota_*$. Considering the previous discussion together with the proof of Proposition 35, we can look at each component at once and follow [KT23b, Proposition 4.30].

Proposition 36. *The composition map c sends $e_j \in \mathbb{Z}^n$ to $((j,j),e)$ with $e \in \pi_1(M)$ being the trivial group element.*

Proof. Let us consider the j-th component in the product fibre bundle. As before, this is just the fibre bundle $\mathbb{S}^3 \xrightarrow{\iota} \mathbb{S}TM \xrightarrow{p} M$. This is locally trivial, for each interior point $x \in M$, there exists a neighbourhood $U \cong \mathbb{D}^4$ around x such that $p^{-1}(U) \cong U \times \mathbb{S}^3 \cong \mathbb{D}^4 \times \mathbb{S}^3$. Points on the boundary work the same but one takes the half-plane. The inclusion $\iota : \mathbb{S}^3 \longrightarrow \mathbb{S}TM$ then factors as $\iota : \mathbb{S}^3 \longrightarrow \mathbb{D}^4 \times \mathbb{S}^3 \longrightarrow \mathbb{S}TM$. Applying the loop space Ω preserves fibration sequences, and we obtain the following factorisation.

$$\begin{tikzcd}
\mathbb{Z}^n \cong \prod_{i=1}^n \pi_2(\Omega\mathbb{S}^3) \arrow[rr, tail, "\iota_*"] \arrow[dr, "\cong"] & & \prod_{i=1}^n \pi_2(\Omega\mathbb{S}TM) \cong \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n,M}), u) \\
& \prod_{i=1}^n \pi_2(\Omega(\mathbb{S}^3 \times \mathbb{D}^4)) \arrow[ur, tail] &
\end{tikzcd}$$

This way, we can assume $M = \mathbb{D}^4$ for the rest of the proof. Note that in this case, both ι_* and Dax are isomorphisms. By definition, $\iota_*(e_j)$ is given by the homotopy class of a a map $F : \mathbb{S}^2 \longrightarrow \mathrm{Imm}_\partial(\mathbb{D}^{1,n}, \mathbb{D}^4) \cong \prod_{i=1}^n \mathrm{Imm}_{\partial_i}(\mathbb{D}^1, \mathbb{D}^4)$ such that the composition

$$\mathcal{D}_u \circ F : \mathbb{S}^2 \xrightarrow{F} \mathrm{Imm}_\partial(\mathbb{D}^{1,n}, \mathbb{D}^4) \xrightarrow{\mathcal{D}_u} \prod_{i=1}^n \Omega(\mathbb{S}^3 \times \mathbb{D}^4)$$

generates exactly the j-th component of $\pi_2(\prod_{i=1}^n \Omega(\mathbb{S}^3 \times \mathbb{D}^4)) \cong \prod_{i=1}^n \pi_2(\Omega(\mathbb{S}^3 \times \mathbb{D}^4)) \cong \mathbb{Z}^n$. On homotopy groups, the latter is given by a section of ι_*, which already is an isomorphism. Thus, it suffices to construct a map $F : \mathbb{S}^2 \longrightarrow \mathrm{Imm}_\partial(\mathbb{D}^{1,n}, \mathbb{D}^4)$ based at u, representing an element in $\pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, \mathbb{D}^4), u)$ with image $((j,j),e)$ under the isomorphism Dax. We want to make use of our realisation map $\mathfrak{r} : \mathbb{Z}[\mathbb{T}_n \times \{e\}] \longrightarrow \pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, \mathbb{D}^4), \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, \mathbb{D}^4), u)$ from Theorem 29, but the boundary conditions of $\mathfrak{r}((i,i),e)$ do not match the requirements of F as the part $\mathbb{I} \times \{0\}$ does not get sent to the basepoint u. Therefore, we need to find a family of *isotopies* from $\mathfrak{r}((j,j),e)$ evaluated on $(t,0) \in \mathbb{I} \times \{0\}$ to u. The isotopy described above by "pulling tight" slides

Let us define such a map $F : \mathbb{I}^2 \longrightarrow (\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, \mathbb{D}^4), u)$. For the upper half of the square, we take the realisation map. For the lower half of the square, we now describe the family of isotopies from $F(t, 1/2) = \mathfrak{r}((j,j),e)(t,0)$ to $F(t,0) = u$. Firstly, consider the state $\mathfrak{r}((j,j),e)(1/2,0)$ as depicted on the left side in Figure 14, once again ignoring the back-and-forth movement along the whiskers, as this is trivial upon taking homotopy classes. We describe the isotopy by "pulling tight" as seen throughout Figure 14.

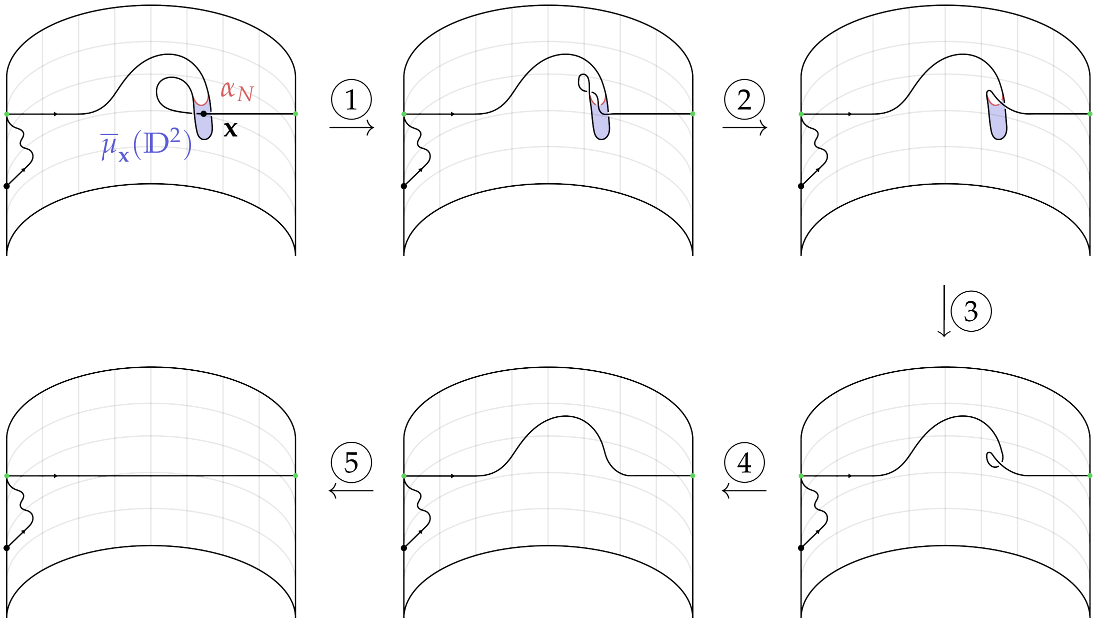


FIGURE 14. Describing the "pulling tight" isotopy to the basepoint u.

Consider $\mathfrak{r}((j,j),e)(\varepsilon,0)$, the state of the arc being pushed into position *before* swinging around the meridian 2-sphere, previously denoted as "pre-loop" position. This is depicted as the state α_N in Figure 14. We compare it to the state $\mathfrak{r}((j,j),e)(1/2,0)$, the arc swung around the sphere half-way. Taking the set-wise symmetric difference of $\mathfrak{r}((j,j),e)(\varepsilon,0)$ and $\mathfrak{r}((j,j),e)(1/2,0)$ yields a 1-sphere which bounds a 2-disk $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^2)$ whose interior intersects the arc only in $\mathbf{x}$. Note that this is precisely the disk seen on the right-hand side of Figure 12, foliated by a family of arcs. We start with an isotopy which drags the part on the left-hand side of $\mathbf{x}$ through $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^2)$, pulling away the point $\mathbf{x}$. This is depicted in ① and ②. Note that this part of the arc naturally winds around a π_1-element by construction. Since we consider the trivial element, this motion is not obstructed and indeed defines an isotopy. The next step is to push the part swung half way around the meridian sphere $\mu_{\mathbf{x}}(\mathbb{S}^2)$ to the state α_N. This can be done by using the guiding disk $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^2)$ since we moved the point $\mathbf{x}$ away earlier. This is depicted in ③. The steps ④ and ⑤ are now obvious isotopies, pushing the arc down into the position of u_j, again using that there is no π_1-element obstructing the motion. Since we only considered the j-th component, and all the other components are in the basepoint configuration by assumption, this yields a path from $\mathfrak{r}((j,j),e)(1/2,0)$ to u through embeddings.

Before we discuss how to obtain a family version of the "pulling tight" isotopy, let us mention why Figure 14 is accurate. Indeed, $\mathbb{D}^4$ is simply-connected and hence we cannot find any π_1-obstruction. It could happen that the j-th arc wraps around another arc u_k in basepoint configuration in order to reach the pre-loop position α_N. In this case, the dimensionally reduced picture does not hold anymore, since we cannot pull the arc through the disk $\overline{\mu}_{\mathbf{x}}(\mathbb{D}^2)$ without intersecting u_k. In dimension four, this works by using a smooth bump function, dipping the arc into the 4th dimension, to circumvent the intersection. This is the same phenomenon as any knot $\mathbb{S}^1 \hookrightarrow \mathbb{D}^4$ being isotopic to the unknot. The alternative, and admittedly much easier, way is to just decrease the radius to obtain the pre-loop position α_N, such that the j-th arc "dips" under the arc u_k.

To obtain a family version without creating any double points (this would change the value of Dax), we taper off the remaining dimension using smooth bump functions. Then we use the analogous isotopy. The only difference is that the 2-disk whose boundary is the set-wise symmetric difference of $\mathfrak{r}((j,j),e)(\varepsilon,0)$ and $\mathfrak{r}((j,j),e)(t,0)$ for some $t \in [0,1]$ does not intersect the arc in the point $\mathbf{x}$. That is not a problem, we can similarly isotope the arc to the basepoint configuration. Hence, we have found a family of isotopies from $\mathfrak{r}((j,j),e)$ to u, not introducing any new double points. The map $F : \mathbb{I}^2 \longrightarrow (\mathrm{Imm}_\partial(\mathbb{D}^{1,n},\mathbb{D}^4),u)$ obtained from gluing the upper and lower half of the square accordingly, only has the double point coming from the realisation map. Hence, applying Dax yields precisely $((j,j),e)$, which had to be shown. ☐

We are now in an excellent position to analyse the diagram stated in the beginning of this section. The discussion will quickly conclude in a proof of Theorem B.

Proof of Theorem B. Consider the following subsets of $\mathbb{T}_n$. The diagonal set $\Delta_n := \{(i,j) \in \mathbb{T}_n : i = j\}$ and the hollow diagonal set $\Delta_n^h := \{(i,j) \in \mathbb{T}_n : i \neq j\}$. We define $\pi_1(M)^{\dagger,n} := \Delta_n \times (\pi_1(M) \setminus \{e\}) \cup \Delta_n^h \times \pi_1(M)$ as the subset of $\mathbb{T}_n \times \pi_1(M)$ that ignores trivial fundamental group elements on the diagonal.

Remark 37. The reader should think about piercing the set $\mathbb{T}_n \times \pi_1(M)$ with a dagger, removing the neutral π_1-element along the diagonal.

With this notation at hand, we have

$$\mathbb{Z}[\pi_1(M)^{\dagger,n}] \cong \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)]/\mathrm{im}(c)$$

and after furthermore comparing the homotopy type of the embedding space to the homotopy type of the smooth mapping space, we obtain the following diagram.

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, "c"] \arrow[d, tail, "\iota_*"] & \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)] \arrow[d, shift left, tail, two heads, "\mathfrak{r}"] & & \\
\pi_2(\mathrm{Imm}_\partial, u) \arrow[r, "\delta_{\mathrm{Imm}}"] \arrow[d, two heads, "p_*"] & \pi_2(\mathrm{Imm}_\partial, \mathrm{Emb}_\partial, u) \arrow[r, "\partial"] \arrow[d, two heads] \arrow[u, shift left, tail, two heads, "\mathsf{Dax}"] & \pi_1(\mathrm{Emb}_\partial, u) \arrow[r, two heads] \arrow[d, equal] & \pi_1(\mathrm{Imm}_\partial, u) \arrow[d, tail, two heads, "p_*"] \\
\pi_2(\mathrm{Map}_\partial, u) \arrow[r, "\delta_{\mathrm{Map}}"] \arrow[d, tail, two heads, "p_u"] & \pi_2(\mathrm{Map}_\partial, \mathrm{Emb}_\partial, u) \arrow[r, "\partial"] \arrow[d, shift left, tail, two heads, "\mathsf{Dax}"] & \pi_1(\mathrm{Emb}_\partial, u) \arrow[r, two heads] & \pi_1(\mathrm{Map}_\partial, u) \arrow[d, tail, two heads, "p_u"] \\
\prod_{i=1}^n \pi_3(M) \arrow[r, dashed, "\mathsf{dax}_u"] & \mathbb{Z}[\pi_1(M)^{\dagger,n})] \arrow[u, shift left, tail, two heads, "\mathfrak{r}"] & & \prod_{i=1}^n \pi_2(M)
\end{tikzcd}$$

The map $\mathsf{dax}_u : \prod_{i=1}^n \pi_3(M) \longrightarrow \mathbb{Z}[\pi_1(M)^{\dagger,n}]$ is defined as the composition $\mathsf{Dax} \circ \delta_{\mathrm{Map}} \circ p_u^{-1}$. Alternatively, one can pick a section s of the surjection p_*, and then consider the composition $\mathsf{Dax} \circ \delta_{\mathrm{Imm}} \circ s \circ p_u^{-1}$. Its image lies in $\mathbb{Z}[\pi_1(M)^{\dagger,n}] \subseteq \mathbb{Z}[\mathbb{T}_n \times \pi_1(M)]$. This leads to a proof of Theorem B. Namely, there is a central group extension

$$\begin{tikzcd}
\mathbb{Z}[\pi_1(M)^{\dagger,n}]/\mathrm{im}(\mathsf{dax}_u) \arrow[r, shift left, tail, "\partial\mathfrak{r}"] & \pi_1(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \arrow[l, shift left, dashed, "\mathsf{Dax}"] \arrow[r, two heads] & \prod_{i=1}^n \pi_2(M).
\end{tikzcd}$$

By the identifications $\pi_1(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), u) \cong \pi_1(\mathrm{Map}_\partial(\mathbb{D}^{1,n}, M), u) \cong \prod_{i=1}^n \pi_2(M)$, the subgroup $\pi_1^D(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$ of $\pi_1(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$, consisting of loops of embeddings that are null-homotopic in the immersion or equivalently mapping space, is isomorphic to the abelian group $\mathbb{Z}[\pi_1(M)^{\dagger,n}]/\mathrm{im}(\mathsf{dax}_u)$. The inclusion map

$$\pi_1^D(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u) \rightarrowtail \pi_1(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$$

is given by $\partial\mathfrak{r}$ which is defined as the point-wise restriction of the realisation map $\mathfrak{r}$ to $\mathbb{I} \times \{0\}$. This amounts to the composition

$$\partial\mathfrak{r}((i_1, i_2), g) : \mathbb{I} \xhookrightarrow{- \times \{0\}} \mathbb{I}^2 \xrightarrow{\mathfrak{r}((i_1,i_2),g)} (\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M)\, \mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$$

and since $\mathfrak{r}((i_1, i_2), g)$ defines an element in the *relative* homotopy group, the image lies entirely in $\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M)$. To give a full description of $\pi_1^D(\mathrm{Emb}_\partial(\mathbb{D}^{1,n}, M), u)$, we need to trace through the definition of dax_u and identify its image.

Remark 38. In the case of $n = 1$, the map dax_u not only appears in the work Kosanović and Teichner but also in the work of Gabai in [Gab21] (with dax_u denoted as d_3 there). Gabai calls the image $\mathrm{im}(\mathsf{dax}_u)$ the "Dax kernel".

Given an element $[f] \in \prod_{i=1}^n \pi_3(M)$, we now describe how to calculate $\mathsf{dax}_u([f])$. Firstly, we consider each component separately, $f_i : (\mathbb{I}^3, \partial I^3) \longrightarrow (M, u_i(-1))$. We remind the reader of Remark 34, stating that the basepoint in the product changes. As $\pi_2(\mathrm{Imm}_\partial(\mathbb{D}^{1,n}, M), u) \cong \prod_{i=1}^n \pi_2(\mathrm{Imm}_{\partial_i}(\mathbb{D}^{1,n}, M), u_i)$ splits as a product, we can lift each f_i to a map $F_i : (\mathbb{I}^2, \partial I^2) \longrightarrow$

$(\mathrm{Imm}_{\partial_i}(\mathbb{D}^1, M), u_i)$ by taking any section of the map $\pi_2(\mathrm{Imm}_{\partial_i}(\mathbb{D}^1, M), u_i) \twoheadrightarrow \pi_3(M, u_i(-1))$. Namely, for $[f_i] \in \pi_3(M, u_i(-1))$, we consider elements $L_{f_i} : (\mathbb{I}^2, \partial I^2) \longrightarrow (\mathrm{Imm}_{\partial_i}(\mathbb{D}^1, M))$ and $L_{u_i} : (\mathbb{I}^2, \partial I^2) \longrightarrow (\mathrm{Imm}_{\partial_i}(\mathbb{D}^1, M))$ with $L_{u_i}(\vec{t}) = u_i$ such that the adjoint of $L_{f_i} \cup_{\partial I^2} L_{u_i}$ represents the element $[f_i]$. Then one computes the geometric Dax-invariant and $\mathsf{dax}_u([f_i])$ is characterised by the equation

$$\mathsf{Dax}([F_i]) = \mathscr{N}(F_i) + \mathsf{dax}_u([f_i])$$

with $\mathscr{N}(F_i)$ being the trivial group elements. Thus, $\mathsf{dax}_u([f_i])$ are exactly the non-trivial group elements under the image of Dax. By additivity of Dax, we have

$$\mathsf{dax}_u([f]) = \textstyle\sum_{i=1}^{n} \mathsf{dax}_u([f_i]) \in \mathbb{Z}[\pi_1(M)^{\dagger,n}].$$

Example 39. The short exact sequence appearing in Theorem B detects the following phenomenon. Let M be a simply-connected 4-manifold M with $\pi_3(M)$ (or vanishing dax_u), and let $n = 1$. In this case, by the exact sequence, there is an isomorphism $\pi_1(\mathrm{Emb}_\partial(\mathbb{D}^1, M), u) \cong \pi_1(\mathrm{Map}_\partial(\mathbb{D}^1, M), u)$. For each homotopy class of a 1-parameter loop of maps from $\mathbb{D}^1$ into M, there is exactly *one* isotopy class of a 1-parameter loop of embeddings from $\mathbb{D}^1$ into M. Now suppose $n = 2$. Then the exact sequence reduces to

$$\mathbb{Z} \rightarrowtail \pi_1(\mathrm{Emb}_\partial(\mathbb{D}^{1,2}, M), u) \longrightarrow\!\!\!\!\!\rightarrow \pi_1(\mathrm{Map}_\partial(\mathbb{D}^{1,2}), u)$$

and for each homotopy class of a 1-parameter loop of maps from $\mathbb{D}^{1,2}$ into M, there are *countably* many isotopy classes of a 1-parameter loop of embeddings from $\mathbb{D}^{1,2}$ into M. In particular, the last sentence holds for *every* 4-manifold M.

V. FORGETTING AUGMENTATIONS

In this section we tie up loose ends from Section II. Namely, for a d-dimensional, compact, oriented, connected manifold M, we study the map

$$\mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M) \xrightarrow{\mathsf{ev}_0} \mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M)$$

forgetting the ε-augmentation. We refer the reader to Section II for the definition of ε-augmented embedding spaces. We will see that we obtain a fibration sequence which allows us to split off a $\mathbb{Z}^n$-factor on the relevant homotopy groups.

V.1. A FIBRATION SEQUENCE

The way one should think about the added datum coming from the ε-augmentation is in the form of a normal vector field of the multi-disk $\mathbb{D}^{k,n}$. In that formulation, it comes to no surprise that it is similar to the case of $n = 1$ which has been dealt with by Kosanović and Teichner in [KT23b, Section 5]. We formalise this observation. Let us fix a Riemannian metric on an oriented, connected, compact d-manifold M. As always, we assume that M has non-empty boundary. The choice of the metric is unique up to homotopy, as the space of Riemannian metrics is known to be convex, and hence contractible as soon as it is non-empty. It is non-empty because Riemannian metrics exist locally on charts. Consider the space $\mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{k,n}, M)$ of *neat* embeddings together with a normal vector field. Furthermore, assume that ε is smaller than the injectivity radius of the metric on M by compactness.

Proposition 40. *The embedding spaces* $\mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M)$ *and* $\mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{k,n}, M)$ *are homotopy equivalent.*

Proof. The proof is the same as the one of [KT23b, Proposition 5.1] which is the case $n = 1$. Let us consider the map

$$\mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M) \xrightarrow{\mathcal{D}^{\uparrow}} \mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{k,n}, M)$$

which is given by sending an embedding of $\mathbb{D}^{k,n} \times [0, \varepsilon]$ to the restricted embedding $\mathbb{D}^{k,n} \times \{0\}$ together with the unit derivative in direction of $[0, \varepsilon]$ as a normal vector field. By Theorem 4, both maps

- $\mathsf{ev}_0 : \mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M) \longrightarrow \mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M)$ and
- $\mathsf{proj}^{\uparrow} : \mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{k,n}, M) \longrightarrow \mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M)$

are fibrations. The unit derivative map $\mathcal{D}^{\uparrow}$ yields a commutative diagram of fibration sequences

$$\begin{tikzcd}
\mathsf{ev}_0^{-1}(u) \arrow[r] \arrow[d] & \mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M) \arrow[r, "\mathsf{ev}_0"] \arrow[d, "\mathcal{D}^{\uparrow}"] & \mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M) \arrow[d, equal] \\
{\mathsf{proj}^{\uparrow}}^{-1}(u^{\uparrow}) \arrow[r] & \mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{k,n}, M) \arrow[r, "\mathsf{proj}^{\uparrow}"] & \mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M)
\end{tikzcd}$$

and we show that the fibres agree up to homotopy equivalence via $\mathcal{D}^{\uparrow}$ restricted on fibres. In that case, the unit derivative map $\mathcal{D}^{\uparrow} : \mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M) \longrightarrow \mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{k}, M)$ is a homotopy equivalence. We start with identifying ${\mathsf{proj}^{\uparrow}}^{-1}(u^{\uparrow})$. Let $u^{\uparrow} := \mathcal{D}^{\uparrow}(u^{\varepsilon})$ be the basepoint in the lower-right embedding space of the diagram. By definition, ${\mathsf{proj}^{\uparrow}}^{-1}(u^{\uparrow})$ is the space of normal vector bundles that agree with $u^{\uparrow}$ on a fixed collar of $\partial\mathbb{D}^{k,n}$. Alternatively, it is the section space $\Gamma_{\partial^{\varepsilon}}(\mathbb{S}\nu u)$ of the sphere bundle $\mathbb{S}\nu u$ agreeing with $u^{\uparrow}$ on the fixed collar of $\mathbb{D}^{k,n}$. We give a homotopy inverse to $\mathbb{D}^{\uparrow}$ restricted to the fibres via the exponential map on the normal bundle. This relates the normal bundle νu to a sufficiently small open disk bundle of u in M, a tubular neighbourhood. Given a unit normal vector field $\xi \in \Gamma_{\partial^{\varepsilon}}(\mathbb{S}\nu u)$, we define the embedding $\mathrm{Exp}_u(\xi) : \mathbb{D}^{k,n} \times [0, \varepsilon] \hookrightarrow M$ as $\mathrm{Exp}_u(\xi)(v, t) := \exp(t \cdot \xi(v))$. This is well-defined since we have assumed the injectivity radius of the metric on M to be larger than ε. Clearly, Exp_u is a right-inverse to $\mathcal{D}^{\uparrow}$ by definition. To show that $\mathcal{D}^{\uparrow}$ and Exp_u are homotopy inverses, we need to construct a homotopy from $\mathrm{Exp}_u \circ \mathcal{D}^{\uparrow}$ to the identity in $\mathsf{ev}_0^{-1}(u)$. Given $\mathscr{E} : \mathbb{D}^{k,n} \times [0, \varepsilon] \hookrightarrow \nu u$, we define $\mathscr{H}_s(v, t) := \frac{\mathscr{E}(v, s \cdot t)}{s}$ for $s \in \mathbb{I}$. This way, we obtain $\mathscr{H}_1 = \mathscr{E}$, and $\mathscr{H}_0$ is the normal derivative of $\mathscr{E}$ at $(v, 0)$. □

Lemma 41. *The section space $\Gamma_{\partial^{\varepsilon}}(\mathbb{S}\nu u)$ is homotopy equivalent to $\prod_{i=1}^{n} \Omega^k \mathbb{S}^{d-k-1}$.*

Proof. Let us consider a trivialisation $\mathbb{S}\nu u \cong \mathbb{D}^{k,n} \times \mathbb{S}^{d-k-1}$ of the sphere bundle of the normal bundle of u in M. On the section space $\Gamma(\mathbb{S}\nu u)$ (without any boundary restrictions), this induces a homeomorphism onto the mapping space $\mathrm{Map}(\mathbb{D}^{k,n}, \mathbb{S}^{d-k-1}) \cong \prod_{i=1}^{n} \mathrm{Map}(\mathbb{D}^{k}, \mathbb{S}^{d-k-1})$ as in the beginning of Section IV. The subspace $\Gamma_{\partial^{\varepsilon}}(\mathbb{S}\nu u) \subseteq \Gamma(\mathbb{S}\nu u)$ is the space of sections that agree with $u^{\uparrow}$ on ∂^{ε}, hence under the identification with $\mathrm{Map}(\mathbb{D}^{k,n}, \mathbb{S}^{d-k-1})$, we obtain

$$\Gamma_{\partial^{\varepsilon}}(\mathbb{S}\nu u) \cong \mathrm{Map}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, \mathbb{S}^{d-k-1}) \cong \prod_{i=1}^{n} \mathrm{Map}_{\partial^{\varepsilon}_i}(\mathbb{D}^{k}, \mathbb{S}^{d-k-1}) \simeq \prod_{i=1}^{n} \Omega^k \mathbb{S}^{d-k-1}$$

after applying Lemma 30 in the form of Remark 31. □

Proposition 40 together with Lemma 41 yields a fibration sequence

$$\textstyle\prod_{i=1}^{n} \Omega^k \mathbb{S}^{d-k-1} \longrightarrow \mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M) \xrightarrow{\mathsf{ev}_0} \mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{k,n}, M).$$

From now on, we go back to the setting of arcs in a 4-manifold. In this case, the fibre space is $\prod_{i=1}^{n} \Omega\mathbb{S}^2$, giving the fibration sequence

$$\textstyle\prod_{i=1}^{n} \Omega\mathbb{S}^2 \longrightarrow \mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M) \longrightarrow \mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M).$$

The inclusion $\mathrm{Emb}_{\partial^\varepsilon}(\mathbb{D}^{1,n}, M) \hookrightarrow \mathrm{Imm}_{\partial^\varepsilon}(\mathbb{D}^{1,n}, M)$ induces a map

$$\mathcal{D}_{u^\varepsilon}\, \mathrm{Emb}^{\uparrow}_{\partial^\varepsilon}(\mathbb{D}^{1,k}, M) \longrightarrow \mathrm{Map}_{\partial}(\mathbb{D}^{1,n}, V_2(M)) \simeq \textstyle\prod_{i=1}^{n} \Omega V_2(M)$$

via the Hirsch-Smale derivative, such that the diagram of fibration sequences

$$\begin{tikzcd}
\prod_{i=1}^{n} \Omega \mathbb{S}^2 \arrow[r] \arrow[d, equal] & \mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{1,n}, M) \arrow[r, "\mathsf{ev}_0"] \arrow[d, "\mathcal{D}_{u^\varepsilon}"] & \mathrm{Emb}_{\partial^\varepsilon}(\mathbb{D}^{1,n}, M) \arrow[d, hook] \\
\prod_{i=1}^{n} \Omega \mathbb{S}^2 \arrow[r] & \prod_{i=1}^{n} \Omega V_2(M)) \arrow[r] & \mathrm{Imm}_{\partial^\varepsilon}(\mathbb{D}^{1,n}, M)
\end{tikzcd}$$

commutes. Recall, $V_\ell(M)$ is defined as the total space of the orthonormal ℓ-frame bundle $V_\ell(M) := V_\ell(TM) \longrightarrow M$ of the tangent bundle of M. We want to show that the upper sequence yields a short exact sequence that splits after applying the fundamental group. To do so, we show that the lower sequence admits such a splitting.

Firstly, let us confirm that we indeed do get short exact sequences. For this, let us collect a number of observations. The Hirsch-Smale map yields a homotopy equivalence $\mathrm{Imm}_{\partial^\varepsilon}(\mathbb{D}^{1,k}, M) \simeq \prod_{i=1}^{n} \Omega \mathbb{S}TM \cong \prod_{i=1}^{n} \Omega V_1(M)$. Since we deal with a product fibration after applying the loop space functor, we can similarly study the sequence

$$\mathbb{S}^2 \longrightarrow V_2(M) \longrightarrow V_1(M) \cong \mathbb{S}TM.$$

For $V_2(M)$ and $V_1(M)$, there are fibration sequences

- $V_1(\mathbb{R}^4) \longrightarrow V_1(M) \longrightarrow M$
- $V_2(\mathbb{R}^4) \longrightarrow V_2(M) \longrightarrow M$

and we can identify the Stiefel manifold $V_1(\mathbb{R}^4) \cong \mathrm{O}(4)/\mathrm{O}(3)$ with $\mathbb{S}^3$. The Stiefel manifold $V_2(\mathbb{R}^4)$ of orthonormal 2-frames can be canonically identified with the sphere bundle of the tangent bundle of $V_1(\mathbb{R}^4) \cong \mathbb{S}^3$. This is one of the rare occasions for which being in dimension 4 is convenient. Namely, viewing $\mathbb{S}^3 \subseteq \mathbb{H}$ as a subset of the quaternions gives $\mathbb{S}^3$ the structure of a Lie group which can be identified with $SU(2)$. Hence, the tangent bundle of $\mathbb{S}^3$ is trivial, and we obtain the identification $V_2(\mathbb{R}^4) \cong \mathbb{S}^3 \times \mathbb{S}^2$. Combining the two fibration sequences above with the comparison of $V_2(M)$ and $V_1(M)$ yields the following diagram of fibration sequences which trivially commutes.

$$\begin{tikzcd}
\mathbb{S}^2 \arrow[r, "\iota"] \arrow[d] & \mathbb{S}^3 \times \mathbb{S}^2 \arrow[r] \arrow[d] & \mathbb{S}^3 \arrow[d] \\
\mathbb{S}^2 \arrow[r] \arrow[d] & V_2(M) \arrow[r] \arrow[d] & \mathbb{S}TM \arrow[d] \\
* \arrow[r] & M \arrow[r] & M
\end{tikzcd}$$

Let us apply the functor $\pi_2(-)$ on the diagram and extend it horizontally once by considering the long exact sequence of homotopy groups associated to the fibration sequences.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"] \arrow[d, "\neq 0"] & \mathbb{Z} \arrow[r, tail, two heads, "\iota_*"] \arrow[d, tail, two heads] & \mathbb{Z} \arrow[r] \arrow[d, tail] & 0 \arrow[d] \\
\pi_3(\mathbb{S}TM) \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[r, tail] \arrow[d] & \pi_2(V_2(M)) \arrow[r, two heads] \arrow[d, two heads] & \pi_2(\mathbb{S}TM) \arrow[d, tail, two heads, "\mathcal{D}"] \\
\pi_3(M) \arrow[r] & 0 \arrow[r] & \pi_2(M) \arrow[r, tail, two heads] & \pi_2(M)
\end{tikzcd}$$

We reason as follows. The map $\iota_* : \mathbb{Z} \to \mathbb{Z}$ must be an isomorphism since it comes from the inclusion $\iota : \mathbb{S}^2 \hookrightarrow \mathbb{S}^3 \times \mathbb{S}^2$. Hence the preceding connecting map $\mathbb{Z} \longrightarrow \mathbb{Z}$ must be 0 by exactness. The map $\mathbb{Z} \longrightarrow \pi_3(\mathbb{S}TM)$ is clearly non-zero as it is obtained from the fibre-inclusion $\mathbb{S}^3 \hookrightarrow \mathbb{S}TM$ as discussed before. Therefore, the connecting map $\pi_3(\mathbb{S}TM) \longrightarrow \mathbb{Z}$ must be 0 by commutativity of the diagram. This implies that both maps $\mathbb{Z} \longrightarrow \pi_2(V_2(M))$ must be injective. The maps $\pi_2(V_2(M)) \longrightarrow \pi_2(M)$ and $\pi_2(V_2(M)) \longrightarrow \pi_2(\mathbb{S}TM)$ must be surjective since $\pi_1(\mathbb{S}^3) \cong \pi_1(\mathbb{S}^3 \times \mathbb{S}^2) = 0$. They agree by the identification of the Hirsch Smale derivative. This discussion immediately gives rise to a commutative diagram

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, tail] \arrow[d, equal] & \pi_1(\mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M), u^{\varepsilon}) \arrow[r, two heads] \arrow[d, tail, two heads, "\mathcal{D}^{\uparrow}_{*}"] & \pi_1(\mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M), u) \arrow[d, equal] \\
\mathbb{Z}^n \arrow[r, tail] \arrow[d, equal] & \pi_1(\mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{1,n}, M), u^{\uparrow}) \arrow[r, two heads] \arrow[d, two heads, "\mathcal{D}_{u^{\varepsilon}_{*}}"] & \pi_1(\mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M), u) \arrow[d, two heads, "\mathcal{D}_{*}"] \\
\mathbb{Z}^n \arrow[r, tail] & \prod_{i=1}^{n} \pi_2(V_2(M)) \arrow[r, two heads] & \prod_{i=1}^{n} \pi_2(\mathbb{S}TM)
\end{tikzcd}$$

which yields the desired short exact sequences. For the splitting of one of the upper short exact sequences, we need to show that the lower short exact sequence admits a splitting. This is the content of the next subsection.

V.2. SPLITTING HOMOTOPY GROUPS OF ORTHONORMAL FRAME BUNDLES

It is a classical result due to Hirzebruch and Hopf that every *closed* oriented 4-manifold admits a Spin^c-structure. In [VT], Teichner and Vogt extended this result to *all* oriented 4-manifolds, hence dropping the condition of the manifold being closed. We will make use of this fact to obtain a splitting of the short exact sequence

$$\mathbb{Z} \rightarrowtail \pi_2(V_2(M)) \twoheadrightarrow \pi_2(M)$$

via a retraction constructed in [KT23b].

The second Stiefel-Whitney class $\omega_2(M) \in H^2(M; \mathbb{Z}/2)$ of the tangent bundle of M comes from pulling back the unique generator ω_2 of $H^2(BO(4); \mathbb{Z}/2)$ along the classifying map $TM :$

$M \longrightarrow BO(4)$. Equivalently, one can post-compose with the non-trivial map $BO(4) \longrightarrow K(\mathbb{Z}/2, 2)$. Since we assume that M is oriented, its tangent bundle carries a canonical orientation and we can similarly consider maps into $B\mathrm{SO}(4)$.

Definition 42. We define the second *spherical* Stiefel-Whitney class $\omega_2^s(M)$ of M as the image of $\omega_2(M)$ under the canonical map $H^2(M; \mathbb{Z}/2) \longrightarrow \mathrm{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}/2)$ pre-composed with the Hurewicz map $\pi_2(M) \longrightarrow H_2(M; \mathbb{Z})$. Therefore, we view the second spherical Stiefel-Whitney class as a map $\omega_2^s(M) : \pi_2(M) \longrightarrow \mathbb{Z}/2$.

An alternative viewpoint is to consider the fibration sequence $\mathrm{SO}(4) \longrightarrow \mathsf{Fr}(M) \longrightarrow M$. Since $\pi_1(\mathrm{SO}(4)) \cong \mathbb{Z}/2$, the long exact sequence of homotopy groups has a connecting map $\pi_2(M) \longrightarrow \mathbb{Z}/2$. This agrees with the second spherical Stiefel-Whitney class as defined above.

We can easily relate the second spherical Stiefel-Whitney class of M to the second (ordinary) Stiefel-Whitney class of $\widetilde{M}$, the universal covering of M. This is the content of the following lemma.

Lemma 43. *Let M be an oriented, connected smooth 4-manifold. Then the second spherical Stiefel-Whitney class $\omega_2^s(M)$ agrees with $\omega_2(\widetilde{M})$.*

Proof. Note that $\pi_1(\widetilde{M})$ vanishes by construction, hence $H^2(\widetilde{M}; \mathbb{Z}/2) \cong \mathrm{Hom}(H_2(\widetilde{M}; \mathbb{Z}), \mathbb{Z}/2)$ since the Ext^1-term vanishes in the short exact sequence of the universal coefficient theorem. Hence, the second Stiefel-Whitney class $\omega_2(\widetilde{M})$ can be seen as a map $\omega_2(\widetilde{M}) : \pi_2(\widetilde{M}) \longrightarrow \mathbb{Z}/2$ after applying the Hurewicz theorem. In particular the second spherical Stiefel-Whitney class of $\widetilde{M}$ *is* the second Stiefel-Whitney class of $\widetilde{M}$. Let us consider the covering map $p : \widetilde{M} \longrightarrow M$. By naturality of the universal coefficient theorem, we obtain the following commutative diagram of short exact sequences.

$$\begin{tikzcd}
\mathrm{Ext}^1(H_1(M; \mathbb{Z}), \mathbb{Z}/2) \arrow[r, tail] \arrow[d] & H^2(M; \mathbb{Z}/2) \arrow[r, two heads] \arrow[d, "p^*"] & \mathrm{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}/2) \arrow[d, "f \mapsto f \circ p_*"] \\
0 \arrow[r] & H^2(\widetilde{M}; \mathbb{Z}/2) \arrow[r, tail, two heads] & \mathrm{Hom}(H_2(\widetilde{M}; \mathbb{Z}), \mathbb{Z}/2)
\end{tikzcd}$$

Since the Stiefel-Whitney classes are natural, the map p^* maps $\omega_2(M)$ to $\omega_2(\tilde{M})$. As previously discussed, we identify $H_2(\widetilde{M}; \mathbb{Z})$ and $\pi_2(\widetilde{M})$ via the Hurewicz isomorphism, and furthermore $\pi_2(\widetilde{M})$ with $\pi_2(M)$ by the fact that higher homotopy groups behave well with taking the universal cover. By following the surjection on the top row, and post-composing with p_*, we therefore indeed obtain the second spherical Stiefel-Whitney class. Since the right arrow on the bottom row is an isomorphism, this yields the identification $\omega_2(\tilde{M}) = \omega_2^s(M)$. ☐

Remark 44. On a side note, the kernel of the map $p^* : H^2(M; \mathbb{Z}/2) \longrightarrow H^2(\widetilde{M}; \mathbb{Z}/2)$ can be easily identified by considering the Serre spectral sequence associated to the fibration sequence

$$\widetilde{M} \xrightarrow{p} M \xrightarrow{q} K(\pi_1(M), 1).$$

We apply $\mathbb{Z}/2$-coefficients. Namely, there is an edge-homomorphism

$$\begin{tikzcd} H^2(M;\mathbb{Z}/2) \arrow[r, two heads] \arrow[rrr, bend left, "p*"] & E_\infty^{0,2} \arrow[r, tail] & E_2^{0,2} \arrow[r, equal] & H^2(\widetilde{M};\mathbb{Z}/2) \end{tikzcd}$$

that agrees with the map $p^* : H^2(M;\mathbb{Z}/2) \longrightarrow H^2(\widetilde{M};\mathbb{Z}/2)$. To compute the kernel of p^*, we need to determine the kernel of this composition. As the second map is injective, this is simply given by the kernel of $H^2(M;\mathbb{Z}/2) \twoheadrightarrow E_\infty^{0,2}$ which can be read off from the filtration along the anti-diagonal. Since $\widetilde{M}$ is simply-connected, the $(p,1)$-row on the E_2-page vanishes, hence on the E_∞-page as well. Therefore, the filtration on the anti-diagonal is given as $0 \subseteq H^2(K(\pi_1(M),1);\mathbb{Z}/2) \subseteq E_\infty^{0,2}$. We have an isomorphism

$$H^2(M;\mathbb{Z}/2) \cong E_\infty^{0,2}/H^2(K(\pi_1(M),1);\mathbb{Z}/2)$$

and the kernel of the map $p^* : H^2(M;\mathbb{Z}/2) \longrightarrow H^2(\widetilde{M};\mathbb{Z}/2)$ can be identified with the abelian group $H^2(K(\pi_1(M),1);\mathbb{Z}/2)$.

As mentioned before, Teichner and Vogt showed in [VT] that any oriented 4-manifold admits a Spin^c-structure, in particular $\widetilde{M}$ does. This means that $\omega_2(\widetilde{M})$ is in the image of the homomorphism $H^2(\widetilde{M};\mathbb{Z}) \longrightarrow H^2(\widetilde{M};\mathbb{Z}/2)$. The commutative diagram

$$\begin{tikzcd} H^2(\widetilde{M};\mathbb{Z}) \arrow[r] \arrow[d, tail, two heads] & H^2(\widetilde{M};\mathbb{Z}/2) \arrow[d, tail, two heads] \\ \mathrm{Hom}(H_2(\widetilde{M}),\mathbb{Z}) \arrow[r, "\text{mod } 2"] & \mathrm{Hom}(H_2(\widetilde{M};\mathbb{Z}),\mathbb{Z}/2) \end{tikzcd}$$

shows that a given Spin^c-structure of $\widetilde{M}$ results in a lift

$$\begin{tikzcd} \pi_2(M) \arrow[r, "\omega_2^s(M)"] \arrow[d, "c"'] & \mathbb{Z}/2 \\ \mathbb{Z} \arrow[ur, "\text{mod } 2"'] & \end{tikzcd}$$

after applying Lemma 43 and the identifications $\pi_2(M) \cong \pi_2(\widetilde{M}) \cong H_2(\widetilde{M};\mathbb{Z})$ as discussed before. The reason why we discussed spherical Stiefel-Whitney classes is the following proposition.

Proposition 45 ([KT23b, Proposition B.14]). *Splittings $\eta : \pi_2(V_2(M)) \longrightarrow \mathbb{Z}$ are in bijection with integer lifts of the second spherical Stiefel Whitney class $\omega_2^s(M)$.*

Therefore, we always obtain splittings of the short exact sequence

$$\begin{tikzcd} \mathbb{Z} \arrow[r, tail] & \pi_2(V_2(M)) \arrow[r, two heads] \arrow[l, bend right, dashed, "\eta"'] & \pi_2(M). \end{tikzcd}$$

These splittings depend on the choice of the Spin^c-structure of the universal covering $\widetilde{M}$. The fact that a Spin^c-structure on $\widetilde{M}$ induces a splitting is mentioned in [KT23a, Section 2.3] without a proof. A chosen splitting η of this sequence then lifts to a splitting of all horizontal sequences

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, tail] \arrow[d, equal] & \pi_1(\mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M), u^{\varepsilon}) \arrow[r, two heads] \arrow[l, dashed, bend right, "\widetilde{\widetilde{\eta}}^{\times n}"'] \arrow[d, tail, two heads, "\mathcal{D}^{\uparrow}_{*}"] & \pi_1(\mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M), u) \arrow[d, equal] \\
\mathbb{Z}^n \arrow[r, tail] \arrow[d, equal] & \pi_1(\mathrm{Emb}^{\uparrow}_{\partial}(\mathbb{D}^{1,n}, M), u^{\uparrow}) \arrow[r, two heads] \arrow[l, dashed, bend right, "\widetilde{\eta}^{\times n}"'] \arrow[d, two heads, "\mathcal{D}_{u^{\varepsilon}_{*}}"] & \pi_1(\mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M), u) \arrow[d, two heads, "\mathcal{D}_{*}"] \\
\mathbb{Z}^n \arrow[r, tail] & \prod_{i=1}^{n} \pi_2(V_2(M)) \arrow[r, two heads] \arrow[l, dashed, bend right, "\eta^{\times n}"'] & \prod_{i=1}^{n} \pi_2(\mathbb{S}TM)
\end{tikzcd}$$

after identifying $\pi_2(\mathbb{S}TM)$ with $\pi_2(M)$ via the Hirsch-Smale derivative, and passing to products. This way, we can split the $\mathbb{Z}^n$-factor off and obtain an isomorphism

$$\pi_1(\mathrm{Emb}^{\varepsilon}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M), u^{\varepsilon}) \cong \mathbb{Z}^n \times \pi_1(\mathrm{Emb}_{\partial^{\varepsilon}}(\mathbb{D}^{1,n}, M), u).$$

Proof of Theorem C. We will now tie up the loose ends to obtain a proof of Theorem C, classifying isotopy classes of neat multi-disks in 4-manifolds in the presence of a geometric dual link. Namely, let M be a 4-dimensional, compact, oriented, connected manifold with non-empty boundary ∂M. Consider an n-component link $\ell : \mathbb{S}^{1,n} \hookrightarrow \partial M$ together with a geometric dual link $G : \mathbb{S}^{2,n} \hookrightarrow \partial M$. We recall the results we have collected so far. In the setting with a dual, Theorem A yields an isomorphism on homotopy groups

$$\pi_0(\mathrm{Emb}_{\ell}(\mathbb{D}^{2,n}, M), \mathrm{U}) \xrightarrow[\cong]{\text{Lemma 9}} \pi_0(\mathrm{Emb}_{\ell^{\varepsilon}}(\mathbb{D}^{2,n}, M), \mathrm{U}) \xrightarrow[\cong]{\text{Theorem A}} \pi_1(\mathrm{Emb}^{\varepsilon}_{u_0^{\varepsilon}}(\mathbb{D}^{1,n}, M_G), u^{\varepsilon}).$$

The manifold M_G is obtained from M by attaching a collection $h^{3,n}$ of 3-handles to the geometric dual link. Furthermore, the observations from the preceding subsections on forgetting the ε-augmentation together with Lemma 9 yields an isomorphisms

$$\pi_1(\mathrm{Emb}^{\varepsilon}_{u_0^{\varepsilon}}(\mathbb{D}^{1,n}, M_G), u^{\varepsilon}) \xrightarrow{\cong} \mathbb{Z}^n \times \pi_1(\mathrm{Emb}_{u_0}(\mathbb{D}^{1,n}, M_G), u).$$

Combining this discussion with Theorem B, we obtain a short exact sequence of sets

$$\begin{tikzcd}
\mathbb{Z}[\pi_1(M_G)^{\dagger,n}]/\mathsf{dax}_u(M_G) \arrow[r, tail, shift left, "\partial\mathfrak{r}"] & \pi_0(\mathrm{Emb}_{\ell}(\mathbb{D}^{2,n}, M), \mathrm{U}) \arrow[r, two heads] \arrow[l, dashed, shift left, "\mathsf{Dax}"] & \mathbb{Z}^n \times \prod_{i=1}^{n} \pi_2(M_G).
\end{tikzcd}$$

Note that $\pi_0(\mathrm{Emb}_{\ell}(\mathbb{D}^{2,n}, M), \mathrm{U})$ can be endowed with a group structure via the delooping coming from Theorem A.

REFERENCES

[BW18] P. BOAVIDA DE BRITO and M. WEISS. Spaces of smooth embeddings and configuration categories. *Journal of Topology* 11.1 (2018), pp. 65–143. DOI: `https://doi.org/10.1112/topo.12048`. URL: `https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/topo.12048`.

[BW13] P. BOAVIDO DE BRITO and M. WEISS. Manifold calculus and homotopy sheaves. *Homology, Homotopy and Applications* 15.2 (2013), pp. 361–383.

[BT82] R. BOTT and L. W. TU. *Differential Forms in Algebraic Topology*. Springer New York, 1982.

[Cer61] J. CERF. Topologie de certains espaces de plongements. fr. *Bulletin de la Société Mathématique de France* 89 (1961), pp. 227–380. DOI: `10.24033/bsmf.1567`. URL: `http://www.numdam.org/articles/10.24033/bsmf.1567/`.

[Cer62] J. CERF. Théorèmes de fibration des espaces de plongements. Applications. fr. *Séminaire Henri Cartan* 15 (1962–1963). talk:8. URL: `http://www.numdam.org/item/SHC_1962-1963__15__A2_0/`.

[Dax72] J.-P. DAX. Étude homotopique des espaces de plongements. fr. *Annales scientifiques de l'École Normale Supérieure* 4e série, 5.2 (1972), pp. 303–377. DOI: `10.24033/asens.1230`. URL: `http://www.numdam.org/articles/10.24033/asens.1230/`.

[Gab20] D. GABAI. The 4-dimensional light bulb theorem. *J. Amer. Math. Soc.* 33.3 (2020), pp. 609–652. ISSN: 0894-0347. DOI: `10.1090/jams/920`.

[Gab21] D. GABAI. Self-referential discs and the light bulb lemma. *Comment. Math. Helv.* 96 (2021), pp. 483–513. DOI: `10.4171/CMH/518`.

[GG73] M. GOLUBITSKY and V. GUILLEMIN. *Stable Mappings and Their Singularities*. Springer New York, NY, 1973. DOI: `https://doi.org/10.1007/978-1-4615-7904-5`.

[Goo90] T. GOODWILLIE. A multiple disjunction lemma for smooth concordance embeddings. *Mem. Amer. Math. Soc.* 86.431 (1990), pp. viii+317. ISSN: 0065-9266. DOI: `10.1090/memo/0431`. URL: `https://doi.org/10.1090/memo/0431`.

[GKK23] T. GOODWILLIE, M. KRANNICH and A. KUPERS. Stability of concordance embeddings. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* (2023), pp. 1–36. DOI: `10.1017/prm.2023.17`.

[GW99] T. GOODWILLIE and M. WEISS. Embeddings from the point of immersion theory : Part II. *Geometry and Topology* 3 (1999), pp. 103–118. DOI: `10.2140/gt.1999.3.103`.

[GKW01] T. G. GOODWILLIE, J. R. KLEIN and M. S. WEISS. Spaces of smooth embeddings, disjunction and surgery. In: *Surveys on surgery theory, Vol. 2*. Vol. 149. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 2001, pp. 221–284.

[Hir59] M. W. HIRSCH. Immersions of Manifolds. *Transactions of the American Mathematical Society* 93.2 (1959), pp. 242–276. ISSN: 00029947. URL: `http://www.jstor.org/stable/1993453`.

[Hir76] M. W. HIRSCH. *Differential Topology*. Springer New York, 1976.

[Hue55] W. HUEBSCH. On the Covering Homotopy Theorem. *Annals of Mathematics* 61.3 (1955), pp. 555–563. ISSN: 0003486X. URL: `http://www.jstor.org/stable/1969813`.

[KK24] B. KNUDSEN and A. KUPERS. Embedding calculus and smooth structures. *Geometry and Topology* 28 (2024), pp. 353–392. DOI: `10.2140/gt.2024.28.353`.

[KT23a] D. Kosanović and P. Teichner. *A new approach to light bulb tricks: Disks in 4-manifolds*. 2023. arXiv: `2209.12015 [math.GT]`.

[KT23b] D. Kosanović and P. Teichner. *A space level light bulb theorem in all dimensions*. 2023. arXiv: `2105.13032 [math.GT]`.

[Mic80] P. Michor. *Manifolds of Differentiable Mappings*. Shiva mathematics series. Shiva Pub., 1980. ISBN: 9780906812037.

[Mil59] J. Milnor. On spaces having the homotopy type of CW-complex. *Trans. Amer. Math. Soc.* 90 (1959), pp. 272–280. DOI: `https://doi.org/10.1090/S0002-9947-1959-0100267-4`.

[Nor69] R. A. Norman. Dehn's Lemma for certain 4-manifolds. *Inventiones mathematicae* 7.2 (1969), pp. 143–147. DOI: `10.1007/BF01389797`. URL: `https://doi.org/10.1007/BF01389797`.

[Pal60] R. S. Palais. Local triviality of the restriction map for embeddings. *Comment. Math. Helv.* 34 (1960), pp. 305–312. DOI: `10.1007/BF02565942`.

[Pal66] R. S. Palais. Homotopy theory of infinite dimensional manifolds. *Topology* 5.1 (1966), pp. 1–16. ISSN: 0040-9383. DOI: `https://doi.org/10.1016/0040-9383(66)90002-4`. URL: `https://www.sciencedirect.com/science/article/pii/0040938366900024`.

[ST22] R. Schneiderman and P. Teichner. Homotopy versus isotopy: Spheres with duals in 4-manifolds. *Duke Mathematical Journal* 171.2 (2022), pp. 273–325. DOI: `10.1215/00127094-2021-0016`. URL: `https://doi.org/10.1215/00127094-2021-0016`.

[Sma58] S. Smale. Regular curves on Riemannian manifolds. *Trans. Amer. Math. Soc.* 87 (1958), pp. 492–512. DOI: `10.2307/1993113`.

[Sma59] S. Smale. The Classification of Immersions of Spheres in Euclidean Spaces. *Annals of Mathematics* 69.2 (1959), pp. 327–344. ISSN: 0003486X. URL: `http://www.jstor.org/stable/1970186`.

[VT] E. Vogt and P. Teichner. *All oriented 4-manifolds have* Spin^c *structures*. URL: `https://people.mpim-bonn.mpg.de/teichner/Math/ewExternalFiles/spin.pdf`.

[Wei99] M. Weiss. Embeddings from the point of immersion theory : Part I. *Geometry and Topology* 3 (1999), pp. 67–101. DOI: `10.2140/gt.1999.3.67`.

[Whi50] J. H. C. Whitehead. A Certain Exact Sequence. *Annals of Mathematics* 52.1 (1950), pp. 51–110. ISSN: 0003486X, 19398980. URL: `http://www.jstor.org/stable/1969511`.