

(STABLE) INFINITY CATEGORIES AND BEYOND

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1. ORGANISATION

This syllabus should be treated as a mere suggestion. Make sure to give intuition, plenty of examples, and provide proofs when there is something to learn from them (e.g. how a previous definition is used in an argument). This is just an outline, and some points may require other definitions or results, which are not explicitly stated here. Furthermore, if you stumble across interesting facts, feel free to simply state those—if you can intuitively explain them, even better.

I will update the syllabus on the go—feel free to suggest any topics you are interested in (they should be suitable for all participants). References for the material include Markus Land’s book on ∞ -categories, Higher Topos Theory and Higher Algebra by Lurie, Kerodon, the ongoing book project by Fabian Hebestreit, as well as Jack Davies’ lecture notes on Algebraic Topology II (which is the course I once took).

As suggested, after each meeting, I may provide one or two exercises, which we may discuss in the following meeting.

2. TENTATIVE SCHEDULE

- (1) *Simplicial sets and infinity categories.* Recall basics on simplicial sets, introduce Kan complexes, as well the definition of an ∞ -category $\mathcal{C}$ (“quasi-category”), and functors between ∞ -categories. Define homotopies in ∞ -categories (2-simplices), and possibly prove that the notion of left and right homotopy agree. Define the homotopy category, and explain commutative diagrams in $\mathcal{C}$ and $\mathbf{hC}$. (**Max**)
- (2) *Kan complexes and nerve.* Explain the difference between Kan complexes and ∞ -categories via a suitable example. State the homotopy hypothesis. Give the construction of the ordinary nerve functor $N: \mathbf{Cat}_1 \rightarrow \mathbf{sSet}$, and identify its essential image with the ∞ -categories which have *unique* inner horn fillings. (**Maxi**)
- (3) *The coherent nerve functor.* Carefully give the construction of the coherent nerve (focus on intuition), and define the infinity category $\mathbf{Spc}$ of spaces, $\mathbf{Cat}_\infty$, and, if time permits, the derived category $\mathcal{D}(R)$ of a commutative ring R (at least explain it intuitively). (**Kamen**)
- (4) *Functor categories.* State and briefly explain the homotopy lifting property for Kan complexes, as well as the analogous description for ∞ -categories (do not even attempt to prove it). Show that the simplicial set $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ refines to an ∞ -category. (**Max**)
- (5) *Mapping spaces.* Define mapping spaces, and show that $\mathbf{Map}_{\mathbf{Nc}}(X, Y)$ is discrete and agrees with $\mathbf{Hom}(X, Y)$. Construct or state the composition map between mapping spaces, prove that composition is unique up to contractible choice. Define what it means for a functor of ∞ -categories to be fully faithful. Define initial and terminal objects in a category, as well as (without stating the details) slice categories. (**Kamen**)
- (6) *Recap I.* Recall the motivation and definition of an ∞ -category, the notion of a homotopy category, a Kan complex and the nerve functor, as well as briefly (!) recall the coherent nerve as well as providing examples of ∞ -categories. (**Maxi**)
- (7) *Recap II.* Recall the homotopy lifting property and related notions, functor categories, as well as mapping spaces. Discuss the notion “contractible choice”, composition, fully faithfulness of functors, as well as briefly define slice categories. (**Kamen**)

- (8) *Limits and colimits in ∞ -categories.* Define limits and colimits in ∞ -categories via slice categories. Begin with products of spaces, and products within an ∞ -categories, then proceed to discuss (homotopy) pullbacks of spaces, and pullbacks in ∞ -categories (for spaces; do not bother with showing that this is independent of the choice of the representative Kan complex). State useful facts about limits/colimits and mapping spaces, functor categories, and that $\mathbf{Spc}$ as well as $\mathbf{Cat}_\infty$ are complete and cocomplete. (**Xiaomeng**)
- (9) *The ∞ -category of spectra.* Discuss the suspension and loop functor on pointed spaces, and define the ∞ -category $\mathbf{Sp}$ of spectra. Unravel the definition to obtain the usual notion of an Ω -spectrum. Define the mapping space in $\mathbf{Sp}$, the suspension and loop functor on spectra (use shift notation), and briefly discuss the adjunction (recall what adjunction means) between these two mutually inverse functors. Introduce the suspension spectrum of a pointed space, the infinite loop space of a spectrum, and prove that the two functors are adjoint. (**Eduardo**)
- (10) *Stable ∞ -categories.* Introduce the notion of stability, and show that $\mathbf{Sp}$ is stable. Introduce the mapping spectrum, and state/outline that stable ∞ -categories are enriched over $\mathbf{Sp}$. Discuss stabilisation of ∞ -categories, and state other interesting examples of stable ∞ -categories (for example $\mathcal{D}^{\mathrm{perf}}(R)$). (**Max**)

From that point, we should discuss the smash product on spectra, homology/cohomology, and perhaps localisation. Afterwards, we may decide where to go. Possible topics are: Spectral sequences via filtered stable categories, algebraic K-theory and the representability result by Ramzi–Sosnilo–Winges, complex-oriented bordism theories and chromatic homotopy theory, functor calculus, ∞ -operads and embedding calculus.

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