POINCARÉ CATEGORIES

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Abstract. These are my (admittedly rough) personal notes of my talk on Poincaré categories given at the European Talbot 2026.

Let R be a commutative ring, and P a projective R-module. An R-valued form on P is simply an element in $\mathrm{Hom}_R(P \otimes_R P, R)$. There is an evident C_2-action on these forms, induced by the C_2-action on $P \otimes_R P$ which flips the variables. Considering the fixed points and orbits of this C_2-action, we obtain

(i) the bijection $\{$*symmetric* R-valued forms on $P\} \longleftrightarrow \mathrm{Hom}_R(P \otimes_R P, R)^{C_2}$, and
(ii) the bijection $\{$*quadratic* R-valued forms on $P\} \longleftrightarrow \mathrm{Hom}_R(P \otimes_R P, R)_{C_2}$.

Symmetric and quadratic forms play a pivotal rôle in classical surgery theory, which is crucial in the classification of high-dimensional manifolds and the study of the homotopy type of the moduli space thereof. Following [CDH+21], we introduce the notion of Poincaré categories, which provide a general framework to obtain homotopical/derived analogues of the above, as well as genuine homotopical versions (there is a difference!).

1. Quadratic functors and hermitian categories

Let $\mathcal{C}$ be a stable ∞-category, and $\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ a reduced functor (i.e. $\Omega(0) = 0$). The composition

$$Ϙ(x) \oplus Ϙ(y) \longrightarrow Ϙ(x \oplus y) \longrightarrow Ϙ(x) \oplus Ϙ(y)$$

agrees with the identity, and so we find a splitting

$$Ϙ(x \oplus y) \simeq Ϙ(x) \oplus Ϙ(y) \oplus \mathrm{B}_Ϙ(x, y).$$

The functor $\mathrm{B}_Ϙ\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is called the *cross effect* of $Ϙ$ (in the sense of Goodwillie calculus, it is the *second* cross effect). The inclusion $\mathrm{B}_Ϙ(x, x) \to Ϙ(x \oplus x)$ refines to a map $\mathrm{B}_Ϙ(x, x)_{\mathrm{h}C_2} \to Ϙ(x)$ upon taking homotopy C_2-orbits with respect to the flip action.

Definition 1.1. A reduced functor $Ϙ\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ is *quadratic* if its cross effect is symmetric bilinear (i.e. preserves exact sequences in both variables), and the cofibre of the canonical map $\mathrm{B}_Ϙ(x, x)_{\mathrm{h}C_2} \to Ϙ(x)$, denoted by $\Lambda_Ϙ$, is linear.

Remark 1.2. This is equivalent to the reduced functor $Ϙ\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ being 2-excisive in the sense of Goodwillie calculus.

Definition 1.3. A Hermitian category $(\mathcal{C}, Ϙ)$ consists of a stable ∞-category $\mathcal{C}$ together with a quadratic functor $Ϙ\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$.

Hermitian categories assemble into an ∞-category $\mathrm{Cat}^{\mathrm{h}}_\infty$ with objects being Hermitian categories, and morphisms between Hermitian categories $(\mathcal{C}, Ϙ)$ and $(\mathcal{C}', Ϙ')$ consisting of

(i) a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{C}'$ between stable ∞-categories, and
(ii) a natural transformation $\eta\colon Ϙ \Rightarrow \mathcal{F}^*Ϙ'$.

Given a stable ∞-category $\mathcal{C}$, we may endow it with different quadratic functors (called *Hermitian structures*). Roughly speaking, to provide a quadratic functor on $\mathcal{C}$, we ought to give a symmetric

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bilinear and a linear part, together with the datum of how they "glue" together. That is, Hermitian structures $\mathrm{Fun}^{\mathrm{q}}(\mathcal{C})$ on $\mathcal{C}$ are classified by the pullback square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{q}}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{B}^{\mathrm{h}C_2}_{(-)}"'] & \mathrm{Ar}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{Fun}^{\mathrm{s}}(\mathcal{C}) \arrow[r, "(\Delta^*(-))^{\mathrm{t}C_2}"] & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}).
\end{tikzcd}$$

Here, $\mathrm{Fun}^{\mathrm{s}}(\mathcal{C})$ denotes the category of symmetric bilinear functors to spectra, $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ the category of linear functors to spectra, and $\mathrm{Ar}(-)$ denotes the arrow category. The morphism $(\Delta^*(-))^{\mathrm{t}C_2}$ takes a symmetric bilinear functor, pre-composed with the diagonal map $\Delta\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}$, and forms its Tate construction with respect to the homotopy C_2 action given by flipping the variables.

Example 1.4. Let $\mathrm{B}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$ be a symmetric bilinear functor. We define $\Omega^{\mathrm{s}}_{\mathrm{B}}(x) := \mathrm{B}(x,x)^{\mathrm{h}C_2}$ as well as $\Omega^{\mathrm{q}}_{\mathrm{B}}(x) := \mathrm{B}(x,x)_{\mathrm{h}C_2}$, and claim that these define quadratic functors. They are clearly reduced since B is bilinear. In either case, it is easily verified that the cross effect is simply given by the symmetric bilinear functor B. Thus, it is left to verify that the cofibre of the maps $\mathrm{B}(x,x)_{\mathrm{h}C_2} \to \Omega^{\mathrm{s}}_{\mathrm{B}}(x)$ and $\mathrm{B}(x,x)_{\mathrm{h}C_2} \to \Omega^{\mathrm{q}}_{\mathrm{B}}(x)$ is linear. In the first case, this map is the norm map, and we obtain $\mathrm{B}(x,x)^{\mathrm{t}C_2}$ which is linear. In the second case, the map is the identity whose cofibre is 0, which is certainly linear.

Example 1.5. Let R be a commutative ring, and $\mathcal{D}^{\mathrm{p}}(R)$ the perfect derived category of R. Consider the symmetric bilinear functor defined by $\mathrm{B}(x,y) := \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x \otimes^L_R y, R)$. The quadratic functors obtained by applying the previous example serve as homotopical analogues of symmetric and quadratic R-valued forms from the introduction. We write Ω^{s}_R and Ω^{q}_R respectively.

2. Poincaré categories

Definition 2.1. A Poincaré category is a Hermitian category $(\mathcal{C}, \Omega)$ such that there exists a duality functor $\mathrm{D}_\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ with the property that $\mathrm{B}_\Omega(-,y) = \mathrm{map}_{\mathcal{C}}(-, \mathrm{D}_\Omega(y))$.

Note that being Poincaré is a property, not a datum. In this case, the cross effect of Ω is said to be *perfect*. By replacing symmetric bilinear functors $\mathrm{Fun}^{\mathrm{s}}(\mathcal{C})$ by *perfect* symmetric bilinear functors $\mathrm{Fun}^{ps}(\mathcal{C})$, we obtain the classification of Poincaré structures $\mathrm{Fun}^{\mathrm{p}}(\mathcal{C})$, given by the pullback square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{p}}(\mathcal{C}) \arrow[r] \arrow[d, "\mathrm{B}^{\mathrm{h}C_2}_{(-)}"'] & \mathrm{Ar}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})) \arrow[d, "\mathrm{ev}_1"] \\
\mathrm{Fun}^{\mathrm{ps}}(\mathcal{C}) \arrow[r, "(\Delta^*(-))^{\mathrm{t}C_2}"] & \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}).
\end{tikzcd}$$

Example 2.2. Let R be a commutative ring, and consider $\mathcal{D}^{\mathrm{p}}(R)$. The symmetric bilinear functor $\mathrm{B}(x,y) = \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x \otimes^L_R y, R)$ is perfect by "currying":

$$\mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x \otimes^L_R y, R) \simeq \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x, \mathrm{RHom}(y,R)),$$

and so the duality is given by sending an object y to $\mathrm{RHom}(y,R)$.

Example 2.3. Let R be a commutative ring, and consider $\mathcal{D}^{\mathrm{p}}(R)$. We give a family of Poincaré structures with underlying cross effect given by $\mathrm{B}(x,y) = \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x \otimes^L_R y, R)$. Recall the classification of Poincaré structures, which, spelt out, results in the diagram

$$\begin{tikzcd}
\mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x \otimes^L_R x, R)_{\mathrm{h}C_2} \arrow[r] \arrow[d, equal] & \Omega(x) \arrow[r] \arrow[d, "\mathrm{B}_\Omega(-)^{\mathrm{h}C_2}"] & \Lambda_\Omega(x) \arrow[d] \\
\mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x \otimes^L_R x, R)_{\mathrm{h}C_2} \arrow[r] & \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x \otimes^L_R x, R)^{\mathrm{h}C_2} \arrow[r] & \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x \otimes^L_R x, R)^{\mathrm{t}C_2}.
\end{tikzcd}$$

Furthermore, we have equivalences

$$\begin{array}{ccc} \Lambda_{\Omega}(x) & \xrightarrow{\simeq} & \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(-,M) \\ \downarrow & & \downarrow \\ \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x\otimes^L_R x,R)^{\mathrm{t}C_2} & \xrightarrow{\simeq} & \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(-,R^{\mathrm{t}C_2}) \end{array}$$

which are induced by the classification of linear functors $\mathcal{D}^{\mathrm{p}}(R)\to \mathrm{Sp}$. The R-module structure on $R^{\mathrm{t}C_2}$ is obtained via the composition $R\to (R\otimes R)^{\mathrm{t}C_2}\to R^{\mathrm{t}C_2}$ of the Tate diagonal followed by the multiplication map. Thus, to give a Poincaré structure on $\mathcal{D}^{\mathrm{p}}(R)$ whose cross effect is $\mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(-\otimes^L_R -,R)$, we need to provide an R-module M together with a morphism $\alpha\colon M\to R^{\mathrm{t}C_2}$. We give a family of interesting examples.

(i) Let $M=\tau_{\geq k}R^{\mathrm{t}C_2}$ together with the canonical map $\alpha\colon \tau_{\geq k}R^{\mathrm{t}C_2}\to R^{\mathrm{t}C_2}$. We denote the resulting Poincaré structure by $\Omega^{\geq k}_R$, and remark that $\Omega^{\geq -\infty}_R=\Omega^{\mathrm{q}}_R$, and $\Omega^{\geq +\infty}_R=\Omega^{\mathrm{s}}_R$.

(ii) Recall the composition $R\to (R\otimes R)^{\mathrm{t}C_2}\to R^{\mathrm{t}C_2}$ (which is called an "$\mathbb{E}_\infty$-Frobenius"), and denote it by α. Setting $M=R$, we thus obtain the Tate–Frobenius Poincaré structure Ω^{TF}_R.

The first example is of further interest, especially in the cases $k\in\{0,1,2\}$. Let P be a projective module. Then $\Omega^{\mathrm{s}}_R(P)$ and $\Omega^{\mathrm{q}}_R(P)$ recover symmetric and quadratic R-valued forms on P by applying $\pi_0(-)$. However, $\Omega^{\mathrm{s}}_R(P)$ does have possibly non-trivial homotopy groups in negative degrees, whereas $\Omega^{\mathrm{q}}_R(P)$ has possibly non-trivial homotopy groups in positive degrees (see below). Thus, Ω^{s}_R and $\Omega^{\mathrm{q}}_R(P)$ serve as mere homotopical analogues to symmetric and quadratic forms. There are genuine homotopical refinements of symmetric and quadratic forms, given by $\Omega^{\geq 0}_R=:\Omega^{\mathrm{gs}}_R$ (genuine symmetric) and $\Omega^{\geq 2}_R=:\Omega^{\mathrm{gq}}_R$ (genuine quadratic) respectively. They have the following property. For a projective module P, we obtain $\Omega^{\mathrm{gs}}_R(P)=\mathrm{HHom}_R(P\otimes_R P,R)^{C_2}$, and $\Omega^{\mathrm{gq}}_R(P)=\mathrm{HHom}_R(P\otimes_R P,R)_{C_2}$, i.e. the homotopy groups are concentrated in degree zero. This follows from [CDH+21, Lemma 4.2.3], which essentially states that $\mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(P\otimes^L_R P,R)$ is contained in the heart of spectra, as well as an application of the exact sequences

$$\Omega^{\mathrm{q}}_R(x)\longrightarrow \Omega^{\geq k}_R(x)\longrightarrow \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x,\tau_{\geq k}R^{\mathrm{t}C_2})$$

and

$$\Omega^{\geq k}_R(x)\longrightarrow \Omega^{\mathrm{s}}_R(x)\longrightarrow \mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(x,\tau_{\leq k-1}R^{\mathrm{t}C_2}).$$

Namely, $\mathrm{map}_{\mathcal{D}^{\mathrm{p}}(R)}(P\otimes^L_R P,R)$ being in the heart of spectra implies that $\Omega^{\mathrm{s}}_R(P)$ is co-connective, and $\Omega^{\mathrm{q}}_R(P)$ is connective. The two exact sequences further imply that $\Omega^{\geq k}_R(P)$ is both connective and co-connective for $k\in\{0,1,2\}$, thus lies in the heart of spectra as well. Comparing $\pi_0\Omega^{\geq k}_R(P)$ to $\pi_0\Omega^{\mathrm{q}}_R(P)$ and $\pi_0\Omega^{\mathrm{s}}_R(P)$ yields the claim about the genuine variants.

Example 2.4. We remark that if $(\mathcal{C},\Omega)$ is a Poincaré category, so is $(\mathcal{C},\Omega[n])$, where $\Omega[n](x):=\Omega(x)[n]$, i.e. the shifted spectrum.

Remark 2.5. Just as in the case of Hermitian categories, Poincaré categories assemble into a category of Poincaré categories, denoted by $\mathrm{Cat}^{\mathrm{p}}_\infty$. However, it is not a full subcategory of $\mathrm{Cat}^{\mathrm{h}}_\infty$; the morphisms are assumed to be "compatible" with the duality. That is, a morphism between Poincaré categories is a morphism between the underlying Hermitian categories such that the natural transformation $\mathcal{F}\circ \mathrm{D}_{\Omega}\Rightarrow \mathrm{D}_{\Omega'}\circ\mathcal{F}^{\mathrm{op}}$ is an equivalence (see [CDH+21, Lemma 1.2.4] and [CDH+21, Definition 1.2.5] for the definition of the natural transformation).

3. Poincaré objects

Definition 3.1. Let $(\mathcal{C},\Omega)$ be a Hermitian category. A *form* in $(\mathcal{C},\Omega)$ consists of $x\in\mathcal{C}$ as well as a point $q\in\Omega^\infty\Omega(x)$ (equivalently a map $q\colon \mathbb{S}\to\Omega(x)$).

One should think of $\Omega(x)$ providing the spectrum of "forms" on x, with the flavour (e.g. quadratic/symmetric/skew-symmetric/even) of the form being dictated by the definition of Ω. Choosing an element $q\in\Omega^\infty\Omega(x)$ corresponds to choosing a specific form on x.

Definition 3.2. Let $(\mathcal{C}, \Omega)$ be a Poincaré category. A *Poincaré object* in $(\mathcal{C}, \Omega)$ is a form (x, q) such that the induced map $q_\#\colon x \to \mathrm{D}_\Omega(x)$ is an equivalence. The map $q_\#$ is given as the image of q under the composition

$$\Omega^\infty \Omega(x) \longrightarrow \Omega^\infty \Omega(x \oplus x) \longrightarrow \mathrm{map}_{\mathcal{C}}(x, \mathrm{D}_\Omega(x))$$

where the second map is the projection onto the cross effect.

Remark 3.3. One can show that forms and Poincaré objects assemble into animae $\mathrm{Fm}(\mathcal{C}, \Omega)$ and $\mathrm{Pn}(\mathcal{C}, \Omega)$ respectively. Furthermore, there are functors $\mathrm{Fm}\colon \mathrm{Cat}_\infty^{\mathrm{h}} \to \mathrm{An}$ and $\mathrm{Pn}\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{An}$, which are both co-represented by the universal Poincaré structure $(\mathrm{Sp}^{\mathrm{fin}}, \Omega^{\mathrm{u}})$ on finite spectra, where $\Omega^{\mathrm{u}} \coloneqq \Omega_{\mathbb{S}}^{\mathrm{TF}}$.

Example 3.4. Consider the Poincaré category $(\mathcal{D}^{\mathrm{p}}(\mathbb{Z}), \Omega_{\mathbb{Z}}^{\mathrm{s}}[-n])$. Let M be a closed, connected, $\mathrm{H}\mathbb{Z}$-oriented manifold of dimension n, with associated co-chain complex $C^*(M; \mathbb{Z})$, which we may view as an object in $\mathcal{D}^{\mathrm{p}}(\mathbb{Z})$. The form

$$q\colon C^*(M; \mathbb{Z}) \otimes_{\mathbb{Z}}^L C^*(M) \xrightarrow{-\cup-} C^*(M; \mathbb{Z}) \xrightarrow{-\cap[M]} \mathbb{Z}[-n]$$

given by the intersection product induces a map $q_\#\colon C^*(M; \mathbb{Z}) \to C_{*-n}(M; \mathbb{Z})$ to the shifted chain complex. Poincaré duality is equivalent to this map being an equivalence.

4. Hyperbolic and metabolic categories

Definition 4.1. Let $\mathcal{C}$ be a stable ∞-category. The *hyperbolic category* is defined as $\mathrm{Hyp}(\mathcal{C}) = (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \Omega_{\mathrm{hyp}})$ with $\Omega_{\mathrm{hyp}}(x, y) \coloneqq \mathrm{map}_{\mathcal{C}}(x, y)$. It is easy to verify that $\mathrm{Hyp}(\mathcal{C})$ is a Poincaré category with duality $\mathrm{D}_{\mathrm{hyp}}(x, y) = (y, x)$.

The Poincaré objects of $\mathrm{Hyp}(\mathcal{C})$ are particularly simple: The duality $\mathrm{D}_{\mathrm{hyp}}(x, y) = (y, x)$ pins down Poincaré objects to satisfy that the map $q\colon (x, y) \to (y, x)$ is an equivalence. Thus, the anima $\mathrm{Pn}(\mathrm{Hyp}(\mathcal{C}))$ is equivalent to the anima of objects in $\mathcal{C}$ together with equivalences—that is, the core groupoid $\mathcal{C}^\simeq$.

Definition 4.2. Let (x, q) be a Poincaré object in a Poincaré category $(\mathcal{C}, \Omega)$. A *Lagrangian* of (x, q) consists of

(i) a map $f\colon L \to x$,
(ii) a null-homotopy $\eta\colon f^*(q) \sim 0$ in $\Omega^\infty \Omega(L)$, and
(iii) we require that the null-homotopy induces an exact sequence $L \to x \simeq \mathrm{D}_\Omega(x) \longrightarrow \mathrm{D}_\Omega(L)$.

If (x, q) admits a Lagrangian (which is not specified), we call the Poincaré object *metabolic*.

Example 4.3. The example one should keep in mind is that of null-bordisms. If $(M, \partial M)$ is a connected, $\mathrm{H}\mathbb{Z}$-oriented manifold of dimension n, the map $i^*\colon C^*(M) \to C^*(\partial M)$ is a Lagrangian, and we obtain Poincaré Lefschetz duality from the third point (Exercise).

Definition 4.4. Let $(\mathcal{C}, \Omega)$ be a Poincaré category. We define the *metabolic category* as $\mathrm{Met}(\mathcal{C}, \Omega) \coloneqq (\mathrm{Ar}(\mathcal{C}), \Omega_{\mathrm{met}})$, where $\Omega_{\mathrm{met}}(L \to x) \coloneqq \mathrm{fib}(\Omega(x) \to \Omega(L))$. As the fibre of a map of quadratic functors, this defines a quadratic functor, and, in fact, a Poincaré structure with duality given by $\mathrm{D}_{\mathrm{met}}(L \to x) = \mathrm{fib}(\mathrm{D}_\Omega(x) \to \mathrm{D}_\Omega(L)) \to \mathrm{D}_\Omega(x)$.

As with the Poincaré objects of the hyperbolic category, we already know the Poincaré objects of the metabolic category. For Poincaré objects $(L \to x, q)$ we require an induced equivalence $x \simeq \mathrm{D}_\Omega(x)$ and the sequence $L \to \mathrm{D}_\Omega(x) \to \mathrm{D}_\Omega(L)$ to be exact. However, this is precisely the datum of a Lagrangian. Thus, $\mathrm{Pn}(\mathrm{Met}(\mathcal{C}, \Omega))$ is equivalent to the anima of Poincaré objects in $(\mathcal{C}, \Omega)$ together with a *specified* Lagrangian $f\colon L \to x$ (so, there is more data than just the metabolic objects).

5. The algebraic Thom isomorphism

Let $(\mathcal{C}, \Omega)$ be a Poincaré category. We briefly state the algebraic Thom isomorphism, which asserts that there is an equivalence

$$\mathrm{Pn}(\mathrm{Met}(\mathcal{C}, \Omega)) \xrightarrow{\simeq} \mathrm{Fm}(\mathcal{C}, \Omega[-1])$$

of animae. This can be shown via the following fact: The inclusion $i\colon \mathrm{Cat}_\infty^{\mathrm{p}} \to \mathrm{Cat}_\infty^{\mathrm{h}}$ admits a right adjoint r (it also admits a left adjoint), such that $(r \circ i)(\mathcal{C}, \Omega) = \mathrm{Met}(\mathcal{C}, \Omega[1])$. Then, using that $(\mathrm{Sp}^{\mathrm{fin}}, \Omega^{\mathrm{u}})$ co-represents both Fm and Pn, we obtain

$$\begin{aligned}\mathrm{Pn}(\mathrm{Met}(\mathcal{C}, \Omega)) &\simeq \mathrm{Map}_{\mathrm{Cat}_\infty^{\mathrm{p}}}((\mathrm{Sp}^{\mathrm{fin}}, \Omega^{\mathrm{u}}), \mathrm{Met}(\mathcal{C}, \Omega)) \simeq \mathrm{Map}_{\mathrm{Cat}_\infty^{\mathrm{p}}}((\mathrm{Sp}^{\mathrm{fin}}, \Omega^{\mathrm{u}}), (r \circ i)(\mathcal{C}, \Omega[-1])) \\ &\simeq \mathrm{Map}_{\mathrm{Cat}_\infty^{\mathrm{h}}}(i(\mathrm{Sp}^{\mathrm{fin}}, \Omega^{\mathrm{u}}), i(\mathcal{C}, \Omega[-1])) \simeq \mathrm{Fm}(\mathcal{C}, \Omega[-1]).\end{aligned}$$

We give a concrete description of the map: A Poincaré object (x, q) together with a specified Lagrangian $(f\colon L \to x, \eta)$ gets sent to the form consisting of the object $\mathrm{fib}(f)$, together with two null-homotopies of $q|_{\mathrm{fib}(f)}$:

(i) the canonical null-homotopy coming from the fibre sequence $\mathrm{fib}(f) \to L \to x$, and
(ii) the one induced by the null-homotopy $\eta\colon f^*(q) \sim 0$ in $\Omega^\infty \Omega(L)$.

We now apply the slogan "in homotopy theory, if two things are trivial for different reasons, the situation is not trivial any more", and find that the two null-homotopies define a loop in $\Omega(\mathrm{fib}(f))$, and thus an element in $\Omega[-1](\mathrm{fib}(f))$.

References

[CDH+21] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle, *Hermitian K-theory for stable ∞-categories I: Foundations*, 2021. 1, 3

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