Symplectic Representation of the Mapping Class Group

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Abstract

We study the action of $Mod(\Sigma_g)$ on $H_1(\Sigma_g; \mathbb{Z})$, following [FM11]. This is given by the map $\Psi: Mod(\Sigma_g) \to Aut(H_1(\Sigma_g; \mathbb{Z}))$ sending [f] to f_* . It is well defined, as, by homotopy invariance, isotopic maps induce the same map in homology. Functoriality of homology results in homeomorphisms inducing isomorphisms in homology. This yields a representation into the symplectic group $Sp(2g,\mathbb{Z})$.

1 Symplectic Representation

Definition 1.1 (Symplectic structure over \mathbb{R}^{2g}). A symplectic structure on a finite-dimensional vector space V is a skew-symmetric alternating bilinear form ω . We now consider $V = \mathbb{R}^{2g}$. Given a basis of \mathbb{R}^{2g} , one can represent ω as a matrix J which is skew-symmetric, invertible and has only zero entries on the diagonal. We now set $Sp(2g, \mathbb{R}) = \{M \in GL(2g, \mathbb{R}) : MJM^T = J\}$ and similarly $Sp(2g, \mathbb{Z}) = Sp(2g, \mathbb{R}) \cap GL(2g, \mathbb{Z}).$

Remark 1.2. For our use, we mostly consider the matrix

$$J = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix}$$

with I_q , the $g \times g$ identity matrix.

Remark 1.3. One can easily check that J satisfies the needed properties. In the upcoming pages, it will get clear why we have chosen J in such a way.

Remark 1.4. For g = 1 we have $Sp(2,\mathbb{Z}) \cong SL(2,\mathbb{Z})$. In this case, Ψ is an isomorphism.

Definition 1.5 (Symplectic structure on $H_1(\Sigma_g; \mathbb{Z})$). We obtain a symplectic structure on $H_1(\Sigma_g; \mathbb{Z})$ by the Poincaré Duality isomorphism

$$PD: H^1(\Sigma_g; \mathbb{Z}) \xrightarrow{\cong} H_1(\Sigma_g; \mathbb{Z})$$

and the intersection form:

$$Q_{\Sigma_g} : H^1(\Sigma_g; \mathbb{Z}) \otimes H^1(\Sigma_g; \mathbb{Z}) \to \mathbb{Z}$$
$$\alpha \otimes \beta \mapsto \langle \alpha \smile \beta, [\Sigma_g] \rangle$$

Let $(a_1, b_1, \ldots, a_g, b_g)$ be the standard basis of $H_1(\Sigma; \mathbb{Z})$. By Poincaré Duality, there is a corresponding dual basis $(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g)$ of $H_1(\Sigma_g; \mathbb{Z})$ satisfying

- (i) $\langle \alpha_i \smile \beta_j, [\Sigma_g] \rangle = \delta_{ij}$
- (ii) $\langle \alpha_i \smile \alpha_j, [\Sigma_g] \rangle = 0$
- (iii) $\langle \beta_i \smile \beta_j, [\Sigma_g] \rangle = 0$

as the cohomology ring of Σ_g is known to be the following.

$$H^{\bullet}(\Sigma_g; \mathbb{Z}) \cong \frac{\mathbb{Z}[\alpha_1, \beta_1, \dots, \alpha_g, \beta_g]}{\langle \alpha_i \alpha_j, \beta_i \beta_j, \alpha_i \beta_j (i \neq j), \alpha_i \beta_i + \beta_i \alpha_i \rangle}$$

Remark 1.6. The condition $MJM^T = J$ in the definition of $Sp(2g, \mathbb{Z})$ is sufficient in order that M represents an element in $Aut(H_1(\Sigma_g; \mathbb{Z}))$. It yields the condition that an element in $Mod(\Sigma_g)$ must preserve the Poincaré dual of the intersection form.

Remark 1.7. Given the standard basis of $H_1(\Sigma_2; \mathbb{Z})$, see Figure 1, and using the corresponding basis $a_i \sim \alpha_i, b_i \sim \beta_i$ of $H^1(\Sigma_2; \mathbb{Z})$ we obtain the following matrices, depending on the ordering of the basis elements.

| | α_1 | α_2 | β_1 | β_2 | | α_1 | β_1 | α_2 | β_2 |
|------------|------------|------------|-----------|-----------|------------|------------|-----------|------------|-----------|
| α_1 | 0 | 0 | 1 | 0 | α_1 | 0 | 1 | 0 | 0 |
| α_2 | 0 | 0 | 0 | 1 | β_1 | -1 | 0 | 0 | 0 |
| β_1 | -1 | 0 | 0 | 0 | α_2 | 0 | 0 | 0 | 1 |
| β_2 | 0 | -1 | 0 | 0 | β_2 | 0 | 0 | -1 | 0 |

If not otherwise stated, we consider the ordering of the left matrix. In the literature, one might find a different ordering.



Figure 1: Geometric realisation of the standard generators of $H_1(\Sigma_2; \mathbb{Z})$.

There is the obvious representation given by the induced map on homology.

$$\Psi: Mod(\Sigma_g) \to Aut(H_1(\Sigma_g; \mathbb{Z})) \cong GL(2g, \mathbb{Z})$$

2 This Mapping is surjective

Now, we have equipped $H_1(\Sigma_g; \mathbb{Z})$ with a symplectic structure. Since this structure is preserved by the image of Ψ , we immediately get the refined mapping $\Psi : Mod(\Sigma_g) \to Sp(2g, \mathbb{Z})$. We want to prove that this is indeed surjective. To do so, we need the following results.

Lemma 2.1. Let γ be an embedded oriented circle on Σ_g . Then for all $a \in H_1(\Sigma_g)$ we have the following. Let γ_* denote the induced class in $H_1(\Sigma_g; \mathbb{Z})$.

$$\Psi(\tau_{\gamma})(a) = (\tau_{\gamma})_*(a) = a + \langle PD^{-1}(\gamma_*) \smile PD^{-1}(a), [\Sigma_g] \rangle \gamma_*$$

Proof. A geometric proof can be found in [FM11, Proposition 6.3].

Theorem 2.2 (Birman). $Sp(2g,\mathbb{Z})$ is generated by the following matrices.

$$A_i = \begin{bmatrix} I_g & -E_{i,i} \\ 0 & I_g \end{bmatrix}, B_i = \begin{bmatrix} I_g & 0 \\ E_{i,i} & I_g \end{bmatrix}, C_j = \begin{bmatrix} I_g & X_j \\ 0 & I_g \end{bmatrix}$$

for $i \in \{1, \ldots, g\}$ and $j \in \{1, \ldots, g-1\}$. Here, we consider elementary matrices and

$$X_j = -E_{j,j} - E_{j+1,j+1} + E_{j,j+1} + E_{j+1,j}$$

given by sums of elementary matrices. Note, that those matrices are invertible.

Proof. This algebraic result is due to Birman.

Theorem 2.3. The map $\Psi: Mod(\Sigma_g) \to Sp(2g,\mathbb{Z})$ is surjective.

Proof. Consider the family of 3g-1 circles given by $\alpha_i, \beta_i, \gamma_j$ for $i \in \{1, \ldots, g\}$ and $j \in \{1, \ldots, g-1\}$ generating $Mod(\Sigma_g)$.



Figure 2: Generators of $Mod(\Sigma_g)$ as given by Lickorish in [Lic64].

Lemma 2.1 now yields the following result.

$$\Psi(\tau_{\alpha_i}) = A_i^{-1}, \ \Psi(\tau_{\beta_i}) = B_i^{-1}, \ \Psi(\tau_{\gamma_j}) = C_j$$

Thus, $\Psi : Mod(\Sigma_g) \to Sp(2g, \mathbb{Z})$ is surjective. The calculation itself is rather tedious. We give some examples. For that, we focus on Σ_2 with the standard basis of $H_1(\Sigma_2; \mathbb{Z})$ with the ordering chosen in Remark 1.7. Note, that the circles α_i and β_i in the figure above correspond to the geometric realisation of the standard basis.

Consider the Dehn-Twist τ_{a_1} . It acts non-trivially on b_1 only.

| [1 | 0 | 0 | 0 | | [1 | 0 | 1 | 0 | | 1 | 0 | 0 | 0 |
|----|---|---|---|--------------------|----|---|---|---|---------------|----|---|---|---|
| 0 | 1 | 0 | 0 | $\Psi(\tau_{a_1})$ | 0 | 1 | 0 | 0 | -1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | \longrightarrow | 0 | 0 | 1 | 0 | \rightarrow | -1 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 1 |

Consider the Dehn-Twist τ_{b_1} . It acts non-trivially on a_1 only.

| [1 | 0 | 0 | 0 | | 1 | 0 | 0 | 0 | | 1 | 0 | 1 | 0 |
|----|---|---|---|--------------------|----|---|---|---|---------------|---|---|---|---|
| 0 | 1 | 0 | 0 | $\Psi(\tau_{b_1})$ | 0 | 1 | 0 | 0 | -1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 0 | | -1 | 0 | 1 | 0 | \rightarrow | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 1 |

In both cases, one can easily observe that the matrices are truly the suggested ones.

Now, onto γ_1 . Firstly, we observe that γ_1 can be represented by the homology class given by $a_1 - a_2$. This can be seen in Figure 3.



Figure 3: The circle γ_1 represented by the homology class $a_1 - a_2$.

Now we consider the Dehn-Twist τ_{γ_1} and its action on the homology generators. Firstly, it is clear that τ_{γ_1} acts trivially on both a_1 and a_2 .

| [1 | 0 | 0 | 0 | | [1 | 0 | 1 | -1 |
|----|---|---|---|-------------------------|----|---|----|----|
| 0 | 1 | 0 | 0 | $\Psi(\tau_{\gamma_1})$ | 0 | 1 | -1 | 1 |
| 0 | 0 | 1 | 0 | \rightarrow | 0 | 0 | 1 | 0 |
| 0 | 0 | 0 | 1 | | 0 | 0 | 0 | 1 |

This agrees with C_1 and hence concludes the exemplary proof.

Remark 2.4. A similar proof involving Burkhardt generators of $Sp(2g,\mathbb{Z})$ was given by Burkhardt. This representation can be used to prove that a generating set of $Mod(\Sigma_g)$ consists of at least 2g + 1 elements. This was originally proven by Humphries, see [Hum06], and is well explained by Farb and Margalit, see [FM11, Section 6.3.3].

References

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