The Birman-Craggs Homomorphisms

Maximilian Hans

27th October 2022

Abstract

In 1978, Birman and Craggs published a paper in which they established a collection of homomorphisms $\mathcal{I}(\Sigma_g) \to \mathbb{Z}/2\mathbb{Z}$, called the Birman-Craggs homomorphisms. We follow [BC78], omitting some details and enriching the text where we find it to be appropriate.

1 Preliminaries

The map $\Psi : Mod(\Sigma_g) \to Aut(H_1(\Sigma_g; \mathbb{Z}))$ sending $[f]$ to f_* yields a surjective representation into the symplectic group $Sp(2g, \mathbb{Z})$ as the intersection form is preserved.

Lemma 1.1. *Let $f \in Mod(\Sigma_g)$ and $\Psi(f)$ be given by the following symplectic matrix.*

$$\Psi(f) = \begin{bmatrix} R & S \\ P & Q \end{bmatrix}$$

Then $M_g(f) = H_g \cup_f H_g$ is an integer homology sphere if and only if S is unimodular over $\mathbb{Z}$.

Proof. This lemma can be found in [BC78, Lemma 2], although Birman and Craggs do use a different ordering of the basis elements. A proof is not given, so we provide one. We must only show that $H_1(M_g(f); \mathbb{Z})$ is trivial if and only if S is unimodular, then the claim follows by Poincaré Duality and the Universal Coefficient Theorem. Consider the Mayer-Vietoris sequence for the canonical pushout.

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{\quad i \quad} & H_g \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q_{H_g}} \\ H_g & \xrightarrow[\quad q_{H_g} \quad]{} & M_g(f) \end{array}$$

In the pushout, we include $i : \Sigma_g \hookrightarrow H_g$ in a canonical way. On the level of homology, this is given by the map $i_* : \bigoplus_{j=1}^{2g} \mathbb{Z} \to \bigoplus_{j=1}^{g} \mathbb{Z}$ which is dropping all the generators a_i, as they are trivial in $H_1(H_g; \mathbb{Z})$. As $f \in Mod(\Sigma_g)$, the induced map on homology is an automorphism on $\bigoplus_{j=1}^{2g} \mathbb{Z}$. Abusing notation, we post-compose with the inclusion and obtain the induced map $f_* : \bigoplus_{j=1}^{2g} \mathbb{Z} \to \bigoplus_{j=1}^{g} \mathbb{Z}$. The Mayer-Vietoris sequence goes as follows.

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow H_2(M_g(f);\mathbb{Z}) \longrightarrow \bigoplus_{j=1}^{2g} \mathbb{Z} \longrightarrow \bigoplus_{j=1}^{2g} \mathbb{Z} \longrightarrow H_1(M_g(f);\mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(M_g(f);\mathbb{Z}) \longrightarrow 0$$

Firstly, observe that $H_1(M_g(f);\mathbb{Z})$ is trivial if and only if the map $\bigoplus_{j=1}^{2g} \mathbb{Z} \to \bigoplus_{j=1}^{2g} \mathbb{Z}$ given by $i_* \oplus f_*$ is an isomorphism. We will now study this map. For that, we take the standard basis of $H_1(\Sigma_g;\mathbb{Z})$.

$$a_1 = e_1, \ldots, a_g = e_g,$$
$$b_1 = e_{g+1}, \ldots, b_g = e_{2g}$$

As the generators a_i are being dropped, we only need to consider what happens to the generators b_i. The map i_* leaves the generators b_i invariant. This is a basis for $\bigoplus_{j=g+1}^{2g} \mathbb{Z}$. Matrix multiplication yields the following result.

$$\Psi(f)(b_1) = \begin{bmatrix} s_{11} \\ \vdots \\ s_{1g} \\ q_{11} \\ \vdots \\ q_{1g} \end{bmatrix}, \ \Psi(f)(b_2) = \begin{bmatrix} s_{21} \\ \vdots \\ s_{2g} \\ q_{21} \\ \vdots \\ q_{2g} \end{bmatrix}, \ \ldots, \ \Psi(f)(b_g) = \begin{bmatrix} s_{g1} \\ \vdots \\ s_{gg} \\ q_{g1} \\ \vdots \\ q_{gg} \end{bmatrix}$$

As the last g entries are already covered by the inclusion i_*, the map is an isomorphism if and only if the matrix S is invertible. As we are working over the ring $\mathbb{Z}$, this is given if and only if S has determinant ± 1. This concludes the proof. $\square$

Definition 1.2 (Torelli group)**.** Consider $\mathcal{I}(\Sigma_g) := ker(\Psi)$. This forms a normal subgroup of $Mod(\Sigma_g)$ and we call it the Torelli group of Σ_g. This yields the following short exact sequence.

$$1 \to \mathcal{I}(\Sigma_g) \to Mod(\Sigma_g) \xrightarrow{\Psi} Sp(2g,\mathbb{Z}) \to 1$$

Theorem 1.3. *Let $S^3 = H_g \cup_f H_g$ be a Heegaard splitting, $k \in \mathcal{I}(\Sigma_g)$. Then the closed oriented 3-manifold defined by $M_g(k \circ f) = H_g \cup_{k \circ f} H_g$ is an integer homology sphere.*

Proof. Any Heegaard splitting can be described as a pushout. Naturally, we can apply the Mayer-Vietoris sequence. Since k is an element of the Torelli group, we know that the induced map in homology, k_*, is trivial. This immediately gives us $(k \circ f)_* = k_* \circ f_* = id \circ f_* = f_*$ and the resulting Mayer-Vietoris sequence of both pushouts

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{i} & H_g \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle q_{H_g}} \\ H_g & \xrightarrow[q_{H_g}]{} & S^3 \end{array} \qquad \begin{array}{ccc} \Sigma_g & \xrightarrow[i]{} & H_g \\ {\scriptstyle k\circ f}\downarrow & & \downarrow{\scriptstyle q'_{H_g}} \\ H_g & \xrightarrow{q'_{H_g}} & M_g(k \circ f) \end{array}$$

is the same, thus $H_\bullet(M_g(k \circ f); \mathbb{Z}) \cong H_\bullet(S^3; \mathbb{Z})$. We abuse notation here, as both f and $k \circ f$ map into ∂H_g. $\square$

2 The Birman-Craggs Homomorphisms

Definition 2.1 (Heegaard embedding)**.** Let M be a closed oriented 3-manifold. A Heegaard embedding is a smooth embedding $i : \Sigma_g \to M$ such that the image is a Heegaard surface. This means $i(\Sigma_g)$ splits M into two handlebodies with the embedded Σ_g as its boundary. This gives us a resulting attaching map $f \in Mod(\Sigma_g)$, hence M is given by the Heegaard splitting $M \cong M_g(f) = H_g \cup_f H_g$.

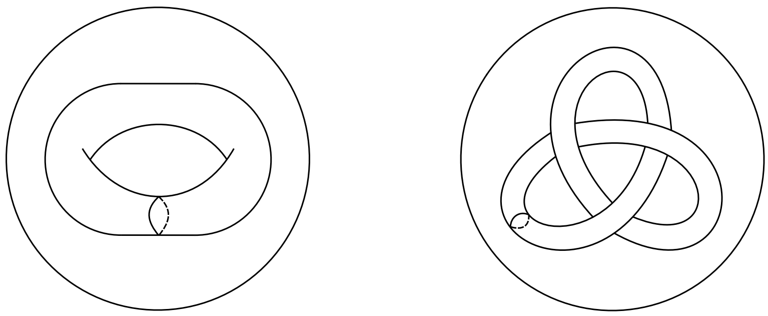

Figure 1: A Heegaard embedding $\Sigma_1 \hookrightarrow S^3$ on the left, an embedding $\Sigma_1 \hookrightarrow S^3$ which is no Heegaard embedding on the right.

Definition 2.2 (Birman-Craggs Homomorphisms)**.** Let $M_g(f)$ be an integer homology sphere obtained from a Heegaard embedding for $g \geq 2$. Define $\rho_f : \mathcal{I}(\Sigma_g) \to \mathbb{Z}/2\mathbb{Z}$ by

$$k \mapsto \mu(M_g(k \circ f)) - \mu(M_g(f)) \mod 2$$

measuring the change in the Rokhlin invariant. We call those maps Birman-Craggs homomorphisms.

At this point, it is everything but obvious that this truly defines a collection of homomorphisms from the Torelli group to $\mathbb{Z}/2\mathbb{Z}$. We will need more results for the proof.

Definition 2.3 (Map pairs and fundamental triple)**.** Let $(f_1, f_2) \in Mod(\Sigma_g) \times Mod(\Sigma_g)$ be a map pair. Each map pair defines a triple of closed oriented 3-manifolds $(M_g(f_2), M_g(f_1), M_g(f_2 \circ f_1^{-1}))$ which we call the fundamental triple for (f_1, f_2).

We now want to associate a map pair with a 4-manifold W such that the following holds.

$$\partial W = -M_g(f_2) \amalg M_g(f_1) \amalg M_g(f_2 \circ f_1^{-1})$$

Definition 2.4. We construct such a manifold. Consider three disjoint copies W_1, W_2, W_3 of the 4-manifold $H_g \times [-1,1]$ and denote $e_j : H_g \times [-1,1] \to W_j$ as the identity maps. We can orientate the manifolds W_j in such a way that $H_g \to H_g \times \{1\} \xrightarrow{e_j} W_j$ is an orientation preserving homeomorphism for $j = 1,2$ and orientation reversing for $j = 3$. We define the following equivalence relation $\sim$ on the 3-manifolds $e_j(\Sigma_g \times [-1,1]) \subseteq W_j$. Let $x \in \Sigma_g$ and $t \in [0,1]$

(i) $e_1(x,t) \sim e_2(f_1(x), -t)$

(ii) $e_1(x,-t) \sim e_3(f_2(x), -t)$

(iii) $e_2(x,t) \sim e_3((f_2 \circ f_1^{-1})(x), t)$

We then set

$$W := \coprod_{j=1}^{3} W_j / \sim$$

and W is an oriented 4-manifold with the desired property. As W is by construction associated with the map pair (f_1, f_2), we write $W(f_1, f_2)$.

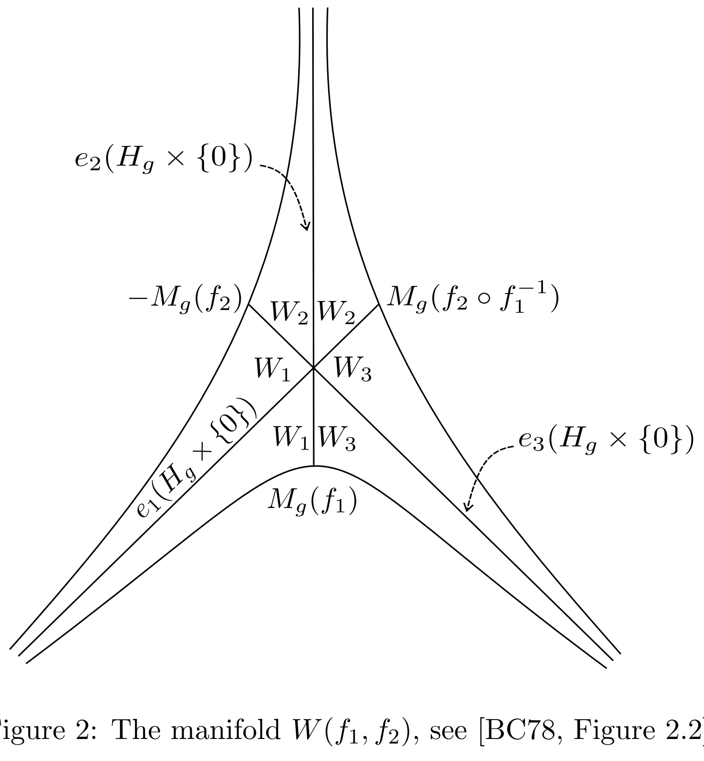

Figure 2: The manifold $W(f_1, f_2)$, see [BC78, Figure 2.2].

We now want to study the intersection form on the oriented 4-manifold $W(f_1, f_2)$.

Lemma 2.5. *Let (f_1, f_2) be a map pair such that the fundamental triple consists of integer homology spheres. Consider $W(f_1, f_2)$. Assume that $Q_{W(f_1,f_2)}$ is even. Then the signature of the intersection form of $Q_{W(f_1,f_2)}$ satisfies the following congruence.*

$$-\mu(M_g(f_2)) + \mu(M_g(f_1)) + \mu(M_g(f_2 \circ f_1^{-1})) \equiv \frac{1}{8}\sigma(W(f_1, f_2)) \mod 2$$

Proof. Let us first ensure that $W(f_1, f_2)$ satisfies the requirements in Rokhlin's Theorem. We observe that $H_1(W; \mathbb{Z}) \cong 0$. This is given by definition, as every boundary component is an integer homology sphere. As the intersection form is even, the manifold is spin. Cutting up $W(f_1, f_2)$ such that

$$\partial W(f_1, f_2) = -M_g(f_2) \amalg M_g(f_1) \amalg M_g(f_2 \circ f_1^{-1})$$

does not change the intersection form and hence leaves the signature invariant. Considering Rokhlin's Theorem and some properties of the Rokhlin invariant we have the following calculation which concludes the proof.

$$\begin{aligned}
&\mu(-M_g(f_2) \amalg M_g(f_1) \amalg M_g(f_2 \circ f_1^{-1})) \\
&= \mu(-M_g(f_2)) + \mu(M_g(f_1)) + \mu(M_g(f_2 \circ f_1^{-1})) \mod 2 \\
&= -\mu(M_g(f_2)) + \mu(M_g(f_1)) + \mu(M_g(f_2 \circ f_1^{-1})) \mod 2 \\
&\equiv \frac{1}{8}\sigma(W(f_1, f_2)) \mod 2
\end{aligned}$$

□

Lemma 2.6. *Let (f_1, f_2) and (f_1', f_2') be both map pairs and consider the induced 4-manifolds $W(f_1, f_2)$ and $W(f_1', f_2')$. Assume that $M_g(f_1)$, $M_g(f_2)$, $M_g(f_1')$ and $M_g(f_2')$ are all integer homology spheres. Then $Q_{W(f_1,f_1)}$ and $Q_{W(f_1',f_2')}$ are equivalent if and only if the map pair $((f_1)_*, (f_2)_*)$ and $((f_1')_*, (f_2')_*)$ is equivalent.*

Proof. As the proof uses other results we do not state, it is omitted, see [BC78, Lemma 6]. □

Lemma 2.7. *Let $f \in Mod(\Sigma_g)$ such that $M_g(f)$ is an integer homology sphere. Then there exists a map pair (f, f') with the following two properties.*

(i) The intersection form of the 4-manifold $W(f, f')$ is even.

(ii) The fundamental triple consists of only integer homology spheres and $M_g(f' \circ f^{-1}) \cong S^3$.

Proof. The proof is constructive and involves the use of the symplectic representation, see [BC78, Lemma 7] □

The following theorem is the main result of this section which is part of [BC78, Theorem 8].

Theorem 2.8. *Consider* $\rho_f : \mathcal{I}(\Sigma_g) \to \mathbb{Z}/2\mathbb{Z}$ *as in Definition 2.2.*

(i) ρ_f *defines a group homomorphism. This is surjective.*

(ii) If f *and* t *both are attaching maps resulting from Heegaard embeddings such that* $\Psi(f) = \Psi(t)$*, then we already have* $\rho_f = \rho_t$*.*

Proof. By assumption, $M_g(f)$ is an integer homology sphere. By applying stabilisation, we may assume that g is even. By Lemma 2.7 we can find a map pair (f, f') such that $M_g(f')$ is an integer homology sphere, $M_g(f' \circ f^{-1}) \cong S^3$ and $Q_{W(f,f')}$ is even. Let us define $t' := f' \circ f^{-1} \circ t$ to obtain the map pair (t, t'). By definition, we have $t' \circ t^{-1} = f' \circ f^{-1}$ and it follows that $\Psi(f') = \Psi(t')$ and $M_g(t' \circ t^{-1}) \cong S^3$. As $M_g(f')$ was assumed to be an integer homology sphere, Lemma 1.1 states that $M_g(t')$ is an integer homology sphere as well. By Lemma 2.6, $Q_{W(t,t')}$ is even. Let $k \in \mathcal{I}(\Sigma_g)$ and consider the map pairs $(k \circ f, f')$ and $(k \circ t, t')$. Note, as k acts trivially on homology, we have $(k \circ f)_* = f_*$ and $(k \circ t)_* = t_*$. Thus, by Lemma 2.6, the homological intersection forms $Q_{W(k \circ f,f')}$ and $Q_{W(f,f')}$ are equivalent, and so are $Q_{W(k \circ t,t')}$ and $Q_{W(t,t')}$, hence their signature is equal. By the signature formula in Lemma 2.5, we have the following.

$$\begin{aligned} &- \mu(M_g(f)) + \mu(M_g(f')) + \mu(M_g(f' \circ f^{-1})) \\ \equiv &- \mu(M_g(k \circ f)) + \mu(M_g(f')) + \mu(M_g(f' \circ f^{-1} \circ k^{-1})) \mod 2 \end{aligned}$$

Similarly, for the pair (t, t').

$$\begin{aligned} &- \mu(M_g(t)) + \mu(M_g(t')) + \mu(M_g(t' \circ t^{-1})) \\ \equiv &- \mu(M_g(k \circ t)) + \mu(M_g(t')) + \mu(M_g(t' \circ t^{-1} \circ k^{-1})) \mod 2 \end{aligned}$$

As $M_g(f' \circ f^{-1}) \cong S^3$, its Rokhlin invariant is 0, and since $f' \circ f^{-1} = t' \circ t^{-1}$ per definition, this simplifies to the following.

$$\begin{aligned} \mu(M_g(k \circ f)) - \mu(M_g(f)) &\equiv \mu(M_g(f' \circ f^{-1} \circ k^{-1})) \mod 2 \\ \mu(M_g(k \circ t)) - \mu(M_g(t)) &\equiv \mu(M_g(f' \circ f^{-1} \circ k^{-1})) \mod 2 \end{aligned}$$

We now have established the equation $\rho_f(k) = \rho_t(k)$. This concludes the proof of the second part. Since $M_g(f' \circ f^{-1}) \cong S^3$ we know by Theorem 1.3 that $M_g(f' \circ f^{-1} \circ k^{-1})$ is an integer homology sphere. Therefore its Rokhlin invariant lies in $\mathbb{Z}/2\mathbb{Z}$. It is left to prove that ρ_f is a group homomorphism. Let $k_1, k_2 \in \mathcal{I}(\Sigma_g)$.

$$\begin{aligned} \rho_f(k_2 \circ k_1) \equiv & \quad \mu(M_g(k_2 \circ k_1 \circ f)) - \mu(M_g(k_1 \circ f)) \\ & + \mu(M_g(k_1 \circ f)) - \mu(M_g(f)) \mod 2 \\ \equiv & \quad \rho_{k_1 \circ f}(k_2) + \rho_f(k_1) \end{aligned}$$

As $\Psi(k_1 \circ f) = \Psi(f)$, by the second assertion, we get $\rho_{k_1 \circ f}(k_2) = \rho_f(k_2)$ and we indeed have the structure of a group homomorphism. This is surjective, given by the Poincaré homology sphere and applying stabilisation. □

Remark 2.9. Consider an arbitrary Heegaard embedding $i : \Sigma_g \to S^3$ with the induced attaching map $f \in Mod(\Sigma_g)$. The Birman-Craggs homomorphism $\rho_f : \mathcal{I}(\Sigma_g) \to \mathbb{Z}/2\mathbb{Z}$ is given by the mapping

$$k \mapsto \mu(M_g(k \circ f))$$

as $\mu(M_g(f)) = \mu(S^3) = 0$.

Remark 2.10. Birman and Craggs proved a slightly more general version of Theorem 2.8. The proof builds upon the special case we have dealt with. In the general version, one does not only consider such homomorphisms obtained from Heegaard embeddings but maps $f_1, f_2 \in Mod(\Sigma_g)$ such that $M_g(f_2 \circ f_1)$ is an integer homology sphere. The Birman-Craggs homomorphisms are then defined as $\rho_{(f_1,f_2)} : \mathcal{I}(\Sigma_g) \to \mathbb{Z}/2\mathbb{Z}$ by the mapping

$$k \mapsto \mu(M_g(f_2 \circ k \circ f_1)) - \mu(M_g(f_2 \circ f_1)) \mod 2$$

measuring the change in the Rokhlin invariant. In 1980, Johnson published a paper in which he proved that every such Birman-Craggs homomorphism is obtained from a Heegaard embedding $i : \Sigma_g \to S^3$ as in the previous remark, see [Joh80, Lemma 7].

We now give some brief outlook on where to go with that result. This will be held informally. In the just mentioned paper, Johnson builds upon the established Birman-Craggs homomorphisms and extends them to a mapping into a certain vector space over $\mathbb{Z}/2\mathbb{Z}$. This combines all possible Birman-Craggs homomorphisms into just one, the Birman-Craggs-Johnson homomorphism. We give a rough outline of its construction, omitting most details and proofs. We follow [Joh80] and [BF07].

Definition 2.11 (Sp-form)**.** An Sp-form is a function $\omega : H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ that satisfies

$$\omega(a + b) = \omega(a) + \omega(b) + a \bullet b$$

with $\bullet : H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) \otimes H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ induced by Poincaré Duality and the cup product evaluated on the generator of $H_2(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$. Note, $\omega(0) = 0$. We set $\Omega(g)$ to be the set of all Sp-forms on $H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$.

We consider the Boolean polynomial algebra $\mathbf{B}(g)$ on $\Omega(g)$. This is defined as the following.

$$\mathbf{B}(g) \coloneqq \frac{\mathbb{Z}/2\mathbb{Z}[\Omega(g)]}{\langle \omega^2 - \omega \rangle}$$

Each homology class $a \in H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$ corresponds to a linear polynomial $P_a \in \mathbf{B}(g)$, given by $\omega \mapsto \omega(a)$.

Lemma 2.12. *The linear polynomials defined above satisfy the following for $a, b \in H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$.*

(i) $P_{a+b} = P_a + P_b + a \bullet b$

(ii) $P_a^2 = P_a$

Proof. The first assertion follows directly from the definition of ω. We have the following calculation.

$$P_{a+b}(\omega) = \omega(a+b) = \omega(a) + \omega(b) + a \bullet b = P_a(\omega) + P_b(\omega) + a \bullet b$$

The second assertion follows from the fact that $\omega(a)^2 = \omega(a)$ as we have $\omega(a) \in \mathbb{Z}/2\mathbb{Z}$. □

Given a symplectic basis $(a_1, b_1, \ldots, a_g, b_g)$ of $H_1(\Sigma_g; \mathbb{Z}/2\mathbb{Z})$, the set of those linear polynomials P_{a_i} and P_{b_i} generates $\mathbf{B}(g)$. We denote $\mathbf{B}_k(g)$ as the subspace of $\mathbf{B}(g)$ which contains polynomials of degree at most k. The polynomial

$$Arf(g) \coloneqq \sum_{i=1}^{g} P_{a_i} P_{b_i}$$

in $\mathbf{B}_2(g)$ does not depend on the choice of the symplectic basis. We now define

$$\tilde{\mathbf{B}}(g) \coloneqq \frac{\mathbf{B}(g)}{Arf(g)}$$

and similarly, denote $\tilde{\mathbf{B}}_k(g)$ to be the subspace of $\tilde{\mathbf{B}}(g)$ which contains polynomials of degree at most k. Johnson showed that all Birman-Craggs homomorphisms can be combined into a homomorphism

$$\sigma : \mathcal{I}(\Sigma_g) \to \tilde{\mathbf{B}}_3(g)$$

called the Birman-Craggs-Johnson homomorphism. This is done by identifying each Heegaard embedding $i : \Sigma_g \to S^3$ with an element $\omega_i \in \Omega(g)$ using self-linking forms. This is fairly well explained by Brendle and Farb, see [BF07, Section 2]. Considering all possible Heegaard embeddings $i : \Sigma_g \to S^3$ and the induced attaching maps $f_i : \Sigma_g \to \Sigma_g$, we get the following result.

$$ker(\sigma) = \bigcap_i ker(\rho_{f_i})$$

The Birman-Craggs-Johnson homomorphism plays a significat role in the evaluation of the homology groups of $\mathcal{I}(\Sigma_g)$. Using the homomorphism $\sigma : \mathcal{I}(\Sigma_g) \to \tilde{\mathbf{B}}_3(g)$, Johnson managed to show that $H_1(\mathcal{I}(\Sigma_g); \mathbb{Z}/2\mathbb{Z}) \cong \tilde{\mathbf{B}}(g)$.

References

[BC78] Joan S. Birman and R. Craggs. 'The μ-Invariant of 3-Manifolds and Certain Structural Properties of the Group of Homeomorphisms of a Closed, Oriented 2-Manifold'. In: *Transactions of the American Mathematical Society* 237 (1978), pp. 283–309.

[Joh80] Dennis Johnson. "Quadratic Forms and the Birman-Craggs Homomorphisms". In: *Transactions of the American Mathematical Society* 261.1 (1980), pp. 235–254.

[BF07] T. E. Brendle and B. Farb. "The Birman–Craggs–Johnson homomorphism and abelian cycles in the Torelli group". In: *Mathematische Annalen* 338 (2007), pp. 33–53.