The Kervaire Invariant and a non-smoothable manifold

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Abstract

Building on Milnor's work on the existence of exotic spheres in [Mil56], Kervaire constructed a manifold that does not admit any smooth structure. The obstruction is given by the Kervaire invariant which can be thought of as a refinement of the mod 2 intersection pairing on an even-dimensional manifold. This short summary serves as a gentle introduction, following Kervaire's original paper [Ker60] and work of Kervaire and Milnor as published in [KM63]. As the theory has evolved since then, we try to give a more general perspective when it is reasonable, enriching the text.

1 Definition of the Kervaire Invariant

The original definition is due to Kervaire as found in [Ker60]. Shortly after its publication, Kervaire and Milnor generalised this definition, which is the one we will focus on for now.

Let M be a closed, $(k-1)$-connected, $2k$-manifold that admits a CW-structure. Since M is simply-connected, it is orientable. By Poincaré duality, the universal coefficient theorem and Hurewicz, it is easy to see that the cohomology of M is concentrated in degree 0, k and $2k$. For any fixed $\alpha \in H^k(M;\mathbb{Z})$, we consider the problem of defining a map $f : M \to \mathbb{S}^k$ such that $f^*(s) = \alpha$, with s generating $H^k(\mathbb{S}^k;\mathbb{Z}) \cong \mathbb{Z}$. Such questions can be answered using obstruction theory. We recall the definition of the obstruction cochain associated to a lifting problem.

Let K be a CW-complex and Y a simple space. Here, simple means that the action of $\pi_1(Y)$ on $\pi_n(Y)$ is trivial. Suppose there exists a map $f : K^{(r)} \to Y$ which we would like to extend to the $(r+1)$-skeleton $K^{(r+1)}$.

Definition 1.1. Associated to any such lifting problem

$$\begin{tikzcd} K^{(r)} \arrow[r, "f"] \arrow[d, hook] & Y \\ K^{(r+1)} \arrow[ur, dashed] & \end{tikzcd}$$

as described above, is the obstruction cochain $c(f) \in C^{r+1}_{\text{cell}}(K;\pi_r(Y))$ defined by

$$c(f)(e^{r+1}) = [f(\partial e^{r+1})] \in \pi_r(Y).$$

Here, e^{r+1} is some $(r+1)$-cell of K. Thus, the element in $\pi_r(Y)$ can be viewed as the composition $S^r \xrightarrow{q} K^{(r)} \xrightarrow{f} Y$ with q the attaching map of the $(r+1)$-cell e^{r+1}. As homotopic gluing maps induce homotopy equivalent spaces, we deal with the set of homotopy classes $[S^r, Y]$. As Y was assumed to be simple, this agrees with the set of pointed homotopy classes $[S^r, Y]_* = \pi_r(Y)$. The main results due to standard obstruction theory are the following.

(1) The map $f : K^{(r)} \to Y$ can be extended to $K^{(r+1)}$ if and only if $c(f) = 0$.

(2) The obstruction cochain $c(f)$ is a cocycle.

The second point makes sure that we can think of $c(f)$ equivalently as an obstruction class $c(f) \in H^{r+1}(K; \pi_r(Y))$. There is the notion of a difference cochain which deals with the choice of different extensions to the $(r+1)$-skeleton of K. As we do not need it, we omit it here.

Back to the problem of defining a map $f : M \to \mathbb{S}^k$ that satisfies $f^*(s) = \alpha$ for a chosen cohomology class $\alpha \in H^k(M; \mathbb{Z})$. By cellular approximation, we can assume that f is trivial on $M^{(k-1)}$. This is backed up by the obstruction theory, as $\pi_r(\mathbb{S}^k)$ vanishes for $r \leq k-1$. As f is assumed to be trivial on $M^{(k-1)}$, any choice of extension on the k-skeleton factors through $M^{(k)}/M^{(k-1)} \cong \bigvee_{I_k} \mathbb{S}^k$. Hence, $f : M^{(k)} \to \mathbb{S}^k$ is entirely characterised by the mapping degree on each sphere. Let $\tilde{\alpha}$ be some cocycle representing the cohomology class α. For a given k-cell e^k, we define

$$f(e^k) \coloneqq \tilde{\alpha}(e^k) \cdot s.$$

Note that $\tilde{\alpha}(e^k) \in \mathbb{Z}$ is the corresponding mapping degree of the map from the sphere obtained by collapsing ∂e^k to $\mathbb{S}^k$. This is given by the factorisation $f : M^{(k)} \to M^{(k)}/M^{(k-1)} \to \mathbb{S}^k$. The obstruction to extend this map to $M^{(k+1)}$ is $c(f) \in C^{k+1}_{\text{cell}}(M; \pi_k(\mathbb{S}^k))$. We calculate

$$c(f)(e^{k+1}) = [f(\partial e^{k+1})] = [\tilde{\alpha}(\partial e^{k+1}) \cdot s] = [\delta\tilde{\alpha}(e^{k+1}) \cdot s] = 0$$

as $\tilde{\alpha}$ is a cocycle representing the chosen cohomology class $\alpha \in H^k(M; \mathbb{Z})$. Suppose we can extend the map f to all of M. Running through the definition of cellular cohomology, it is easy to see that the condition $f^*(s) = \alpha$ is indeed satisfied. The remaining cohomology groups of M vanish until degree $2k$. Hence, the final obstruction cocycle lies in $C^{2k}_{\text{cell}}(M; \pi_{2k-1}(\mathbb{S}^k))$. Equivalently, we can consider the obstruction class in $H^{2k}(M; \pi_{2k-1}(\mathbb{S}^k))$.

Definition 1.2. The Kervaire class $c(\alpha)$ of M is defined to be this final obstruction class. The Kervaire form is the associated map $c : H^k(M; \mathbb{Z}) \to H^{2k}(M; \pi_{2k-1}(\mathbb{S}^k))$.

Lemma 1.3. *The Kervaire form c is natural. Namely, if $g : M \to M'$ is a map between manifolds satisfying the necessary conditions, the following square commutes.*

$$\begin{tikzcd}
H^k(M;\mathbb{Z}) \arrow[d, "c"'] & H^k(M';\mathbb{Z}) \arrow[l, "g^*"'] \arrow[d, "c"] \\
H^{2k}(M;\pi_{2k-1}(\mathbb{S}^k)) & H^{2k}(M';\pi_{2k-1}(\mathbb{S}^k)) \arrow[l, "g^*"']
\end{tikzcd}$$

This immediately implies that the Kervaire form does not depend on the chosen CW-structure.

Lemma 1.4. *The Kervaire form satisfies $c(\alpha + \beta) = c(\alpha) + c(\beta) + [s, s](\alpha \smile \beta)$. In the case of $\pi_{2k-1}(\mathbb{S}^k)$ being a field, as the cup-product pairing is bilinear, this implies that the Kervaire form is a quadratic form.*

Proof. Let U be the space obtained from $\mathbb{S}^k$ by killing all homotopy groups of degree at least $2k-1$. Even though U is not a manifold, it makes sense to talk about its Kervaire class as all the needed properties are fulfilled. Indeed, U is $(k-1)$-connected and its cohomology groups $H^r(U; \mathbb{Z})$ vanishes for $k+1 \leq r \leq 2k-1$. These assumptions hold for the space $U \times U$ as well. This gives rise to the following two forms.

(1) $c : H^k(U;\mathbb{Z}) \to H^{2k}(U;\pi_{2k-1}(\mathbb{S}^k))$

(2) $c : H^k(U \times U;\mathbb{Z}) \to H^{2k}(U \times U;\pi_{2k-1}(\mathbb{S}^k))$

We have $H^k(U;\mathbb{Z}) \cong H^k(\mathbb{S}^k;\mathbb{Z}) \cong \mathbb{Z}$, let u be a generator of $H^k(U;\mathbb{Z})$. Furthermore, by the Kuenneth formula, we have $H^k(U \times U;\mathbb{Z}) \cong H^k(U;\mathbb{Z}) \oplus H^k(U;\mathbb{Z})$ and $H^{2k}(U \times U;\pi_{2k-1}(\mathbb{S}^k)) \cong H^{2k}(U;\pi_{2k-1}(\mathbb{S}^k)) \oplus H^{2k}(U;\pi_{2k-1}(\mathbb{S}^k)) \oplus H^k(U;\pi_{2k-1}(\mathbb{S}^k)) \otimes H^k(U;\pi_{2k-1}(\mathbb{S}^k))$. Using this splitting, we get that

$$c(u \otimes 1 + 1 \otimes u) = a \otimes 1 + 1 \otimes b + \gamma(u \otimes u)$$

for a and b in $H^{2k}(U;\pi_{2k-1}(\mathbb{S}^k))$ and $\gamma \in \pi_{2k-1}(\mathbb{S}^k)$ a coefficient. We claim that $a = c(u) = b$. For this, we consider the inclusion $U \times e^0 \hookrightarrow U \times U$ and apply naturality of c.

$$\begin{tikzcd}
u \arrow[dd, mapsto] & & & u \otimes 1 + 1 \otimes u \arrow[lll, mapsto] \arrow[ddd, mapsto] \\
& H^k(U;\mathbb{Z}) \arrow[d, "c"'] & H^k(U \times U;\mathbb{Z}) \arrow[l, "\mathsf{pr}_1"'] \arrow[d, "c"] & \\
c(u) \arrow[dr, equal] & H^{2k}(U;\pi_{2k-1}(\mathbb{S}^k)) & H^{2k}(U \times U;\pi_{2k-1}(\mathbb{S}^k)) \arrow[l, "\mathsf{pr}_1"'] & \\
& a & & a \otimes 1 + 1 \otimes b + \gamma(u \otimes u) \arrow[ll, mapsto]
\end{tikzcd}$$

Considering the inclusion $e^0 \times U \hookrightarrow U \times U$ yields $b = c(u)$. It is left to evaluate the coefficient $\gamma \in \pi_{2k-1}(\mathbb{S}^k)$. We claim that γ is given by the Whitehead product class $[s, s]$. Consider the inclusion $\mathbb{S}^k \times \mathbb{S}^k \hookrightarrow U \times U$. The splitting of $H^{2k}(U \times U;\pi_{2k-1}(\mathbb{S}^k))$ in the Kuenneth formula comes from a splitting on the $(2k)$-skeleton of $U \times U$. The embedded $\mathbb{S}^k \times \mathbb{S}^k$ contributes to the factor $H^k(U;\pi_{2k-1}(\mathbb{S}^k)) \otimes H^k(U;\pi_{2k-1}(\mathbb{S}^k))$. Applying naturality yields the following commutative diagram.

$$\begin{tikzcd}
s \otimes 1 + 1 \otimes s \arrow[dd, mapsto] & & & u \otimes 1 + 1 \otimes u \arrow[lll, mapsto] \arrow[ddd, mapsto] \\
& H^k(\mathbb{S}^k \times \mathbb{S}^k;\mathbb{Z}) \arrow[d, "c"'] & H^k(U \times U;\mathbb{Z}) \arrow[l, "\cong"'] \arrow[d, "c"] & \\
c(s \otimes 1 + 1 \otimes s) \arrow[dr, equal] & H^{2k}(\mathbb{S}^k \times \mathbb{S}^k;\pi_{2k-1}(\mathbb{S}^k)) & H^{2k}(U \times U;\pi_{2k-1}(\mathbb{S}^k)) \arrow[l, "\mathsf{pr}_3"'] & \\
& \gamma(s \otimes s) & & a \otimes 1 + 1 \otimes b + \gamma(u \otimes u) \arrow[ll, mapsto]
\end{tikzcd}$$

Note that $c(s \otimes 1 + 1 \otimes s)$ is the obstruction class associated to the problem of defining a map $f : \mathbb{S}^k \times \mathbb{S}^k \to \mathbb{S}^k$ such that $f^*(s) = s \otimes 1 + 1 \otimes s$. Applying Hurewicz, this can be rephrased as f satisfying $f \circ i_1 = s$ and $f \circ i_2 = s$. Here, s is represents the identity map $s : \mathbb{S}^k \to \mathbb{S}^k$ as it generates $\pi_k(\mathbb{S}^k) \cong H^k(\mathbb{S}^k)$. We claim that $[s, s] = 0$ if and only if such a map f exists, hence concluding that $\gamma = [s, s]$. Consider the following diagram.

$$\begin{tikzcd}
\mathbb{S}^{2k-1} \arrow[r, "q"] \arrow[d, hook, "j"'] & \mathbb{S}^k \vee \mathbb{S}^k \arrow[r, "s \vee s"] \arrow[d, "i_1 \vee i_2"] & \mathbb{S}^k \\
\mathbb{D}^{2k} \arrow[r, "Q"] \arrow[ur, phantom, "\ulcorner"] & \mathbb{S}^k \times \mathbb{S}^k &
\end{tikzcd}$$

The composition of the upper row is the Whitehead product, the square is the CW-pushout diagram associated to $\mathbb{S}^k \times \mathbb{S}^k$.

(1) Suppose $[s,s]=0$. Therefore, the composition $(s\vee s)\circ q:\mathbb{S}^{2k-1}\to\mathbb{S}^k$ is homotopically trivial, and it extends to a map $\tilde{f}:\mathbb{D}^{2k}\to\mathbb{S}^k$. Applying the universal property of the pushout to the map $s\vee s:\mathbb{S}^k\vee\mathbb{S}^k\to\mathbb{S}^k$ yields the existence of the desired map $f:\mathbb{S}^k\times\mathbb{S}^k\to\mathbb{S}^k$, satisfying the necessary conditions.

(2) Suppose such a map $f:\mathbb{S}^k\times\mathbb{S}^k\to\mathbb{S}^k$ exists. Then $s\vee s=F\circ(i_1\vee i_2)$ and $(s\vee s)\circ q=F\circ(i_1\vee i_2)\circ q$. Since $(i_1\vee i_2)\circ q=Q\circ j$ and $\mathbb{D}^{2k}$ is contractible, the composition $(i_1\vee i_2)\circ q:\mathbb{S}^{2k-1}\to\mathbb{S}^k\times\mathbb{S}^k$ is homotopically trivial. Thus, $[s,s]=0$.

Together, we have shown that $c(u\otimes 1+1\otimes u)=c(u)\otimes 1+1\otimes c(u)+[s,s](u\otimes u)$. Notice that $u\otimes u=(u\otimes 1)\smile(1\otimes u)$. Consider two cohomology classes α and β in $H^k(M;\mathbb{Z})$. Since $\pi_{2k-1}(U)=0$, the higher obstruction classes vanish, and we can define two maps $g_\alpha:M\to U$ and $g_\beta:M\to U$ such that $g_\alpha^*(u)=\alpha$ and $g^*(u)=\beta$. Let us define $g:M\to U\times U$ as (g_α,g_β). It is clear that $g^*(u\otimes 1)=\alpha$ and $g^*(1\otimes u)=\beta$. Using naturality of the Kervaire form, we have

$$\begin{aligned}
c(\alpha+\beta) &= c(g^*(u\otimes 1+1\otimes u))\\
&= g^*(c(u\otimes 1+1\otimes u))\\
&= g^*(c(u)\otimes 1+1\otimes c(u)+[s,s]((u\otimes 1)\smile(1\otimes u)))\\
&= g^*(c(u\otimes 1))+g^*(c(1\otimes u))+[s,s]g^*((u\otimes 1)\smile(1\otimes u))\\
&= c(g^*(u\otimes 1))+c(g^*(1\otimes u))+[s,s](g^*((u\otimes 1))\smile g^*((1\otimes u)))\\
&= c(\alpha)+c(\beta)+[s,s](\alpha\smile\beta).
\end{aligned}$$

This concludes the proof. □

So far, it is not clear why the Kervaire invariant should be $\mathbb{Z}/2$-valued. In the case of $k=5$, we have $\pi_9(\mathbb{S}^5)\cong\mathbb{Z}/2$. The Kervaire form $c:H^5(M;\mathbb{Z})\to\mathbb{Z}/2$ induces a form $\tilde{c}:H^5(M;\mathbb{Z}/2)\to\mathbb{Z}/2$. Indeed, we have $c(2\alpha)=c(\alpha)+c(\alpha)+\alpha\cup\alpha=0$. Furthermore, we have the following result.

Lemma 1.5. *Let k be odd, M a closed, $(k-1)$-connected, framed $2k$-manifold. An embedded k-sphere $\mathbb{S}^k$ in M has trivial normal bundle if and only if its dual cohomology class $\alpha\in H^k(M;\mathbb{Z})$ satisfies $c(\alpha)=0$.*

Proof. Let $\nu:\mathbb{S}^k\to\mathrm{BSO}(k)$ be the normal bundle of the embedded k-sphere in M and $N\subseteq M$ a tubular neighbourhood. By excision and the long exact sequence of pairs, $H^{2k}(\mathrm{Th}(\nu);\pi_{2k-1}(\mathbb{S}^k))\cong H^{2k}(M,M\setminus\dot{N};\pi_{2k-1}(\mathbb{S}^k))\cong H^{2k}(M;\pi_{2k-1}(\mathbb{S}^k))$. Naturality of c yields the commutativity of the following diagram.

$$\begin{array}{ccc}
i_k & \longmapsto & c(i_k)\\
\big\downarrow & \begin{array}{ccc} H^k(\mathrm{Th}(\nu);\mathbb{Z}) & \xrightarrow{c} & H^{2k}(\mathrm{Th}(\nu);\pi_{2k-1}(\mathbb{S}^k))\\ \big\downarrow{\scriptstyle P^*} & & \big\downarrow{\scriptstyle\cong}\\ H^k(M;\mathbb{Z}) & \xrightarrow{c} & H^{2k}(M;\pi_{2k-1}(\mathbb{S}^k)) \end{array} & \big\downarrow\\
\alpha & \longmapsto & c(\alpha)
\end{array}$$

Here, i_k generates $H^k(\mathrm{Th}(\nu);\mathbb{Z})$ and $P:M\to\mathrm{Th}(\nu)$ is the collapse-map. With a chosen isomorphism, we have $c(i_k)=c(\alpha)\in\pi_{2k-1}(\mathbb{S}^k)$. Therefore, the Kervaire class only depends on the embedded sphere.

Since M is framed, ν is stably trivial, hence it lies in the kernel of the map

$$\pi_k(\mathrm{BSO}(k)) \to \pi_k(\mathrm{BSO})$$

which, for k odd and $k \notin \{1,3,7\}$, is cyclic of order 2. Hence, in this case, the Kervaire form is $\mathbb{Z}/2$-valued. The only non-trivial element is given by the tangent bundle $T\mathbb{S}^k$, which can be identified with the normal bundle of the diagonal $\Delta \subseteq \mathbb{S}^k \times \mathbb{S}^k$. Choosing an orientation on $\mathrm{Th}(\nu)$ and $\mathbb{S}^k \times \mathbb{S}^k$ corresponds to the choice of the fundamental class $[\mathrm{Th}(\nu)]$ and $[\mathbb{S}^k \times \mathbb{S}^k]$. Suppose ν is non-trivial, then it can be identified with the normal bundle of $\Delta \subseteq \mathbb{S}^k \times \mathbb{S}^k$. By previous results, we have

$$c(i_k)[\mathrm{Th}(\nu)] = c(s \otimes 1 + 1 \otimes s)[\mathbb{S}^k \times \mathbb{S}^k] = [s,s] \neq 0.$$

If ν is trivial, $c(i_k)$ clearly vanishes. This concludes the proof. $\square$

Similarly as discussed above, in the case of M being framed, the Kervaire form induces a form $\tilde{c}: H^k(M;\mathbb{Z}/2) \to \mathbb{Z}/2$ satisfying the analogue properties.

Definition 1.6. We define the Kervaire invariant $\Phi(M)$ to be the Arf-invariant of $\tilde{c}$. That is, given a symplectic basis $\{\alpha_i, \beta_i\}_{i=1}^n$ of $H^k(M;\mathbb{Z}/2)$, $\Phi(M) \coloneqq \sum_{i=1}^n \tilde{c}(\alpha_i)\tilde{c}(\beta_i)$.

An alternative interpretation of the Kervaire invariant has been given by Browder, see [Bro69]. We recall the definition here. Let M be a closed, framed $2k$-manifold. Then there is an embedding $i: M \hookrightarrow \mathbb{R}^{2k+r}$ such that the normal bundle ν of M admits a trivialisation. Notice that its Thom space $\mathrm{Th}(\nu)$ is then given by $\Sigma^r(M)$. Identifying $\mathbb{S}^{2k+r}$ with $(\mathbb{R}^{2k+r})^+$, the one-point compactification, the Pontrjagin-Thom collapse map is given by $P: \mathbb{S}^{2k+r} \to \Sigma^r(M)$.

Definition 1.7. Let $\alpha \in H^k(M;\mathbb{Z}/2)$ represented by $\alpha: M \to K(\mathbb{Z}/2,k)$. Pre-composition with the Pontrjagin-Thom collapse map as above gives

$$\mathbb{S}^{2k+r} \xrightarrow{P} \Sigma^r(M) \xrightarrow{\Sigma^r(\alpha)} \Sigma^r(K(\mathbb{Z}/2),k)$$

defining an element in $\pi_{2k+r}(\Sigma^r(K(\mathbb{Z}/2,k)))$. As r can be chosen to be arbitrary large, we're dealing with homotopy groups of the suspension spectrum of $K(\mathbb{Z}/2,k)$, namely $\pi_{2k}(\Sigma^\infty(K(\mathbb{Z}/2,k))) \cong \mathbb{Z}/2$. The corresponding element is denoted as $\tilde{c}(\alpha)$, and this defines a quadratic form $\tilde{c}: H^k(M;\mathbb{Z}/2) \to \mathbb{Z}/2$. Its Arf-invariant is the Kervaire invariant.

This definition agrees with the one given by Kervaire on its common domain of definition. Namely, using framed surgery, we can assume that the manifold M is $(k-1)$-connected. We know that, in this case, the Kervaire invariant can be geometrically interpreted as a question about embedded spheres having trivial normal bundle ν in M. Let $\alpha \in H^k(M;\mathbb{Z}/2)$ and consider the normal bundle ν of the representing embedded sphere $\mathbb{S}^k \subseteq M$. Let i_5 be the generator of $H^k(\mathrm{Th}(\nu);\mathbb{Z}/2)$. The map $\alpha: M \to K(\mathbb{Z}/2,k)$ can be factored

$$M \xrightarrow{P} \mathrm{Th}(\nu) \xrightarrow{\tilde{\alpha}} K(\mathbb{Z}/2,k)$$

with $\tilde{\alpha}^*(\kappa) = i_5$, with κ being the generator of $H^k(K(\mathbb{Z}/2,k))$.

2 The Kervaire Manifold

We will focus on the construction of the Kervaire manifold, a manifold that does not admit any smooth structure. The obstruction to show this is going to be the Kervaire invariant. Consider the tangent bundle $T\mathbb{S}^5$ of $\mathbb{S}^5$, and its associated disk bundle which we will denote by $d: E \to \mathbb{S}^5$. Let $\mathbb{D}^5 \subseteq \mathbb{S}^5$ be embedded, such that $d|_{\mathbb{D}^5} \cong \mathbb{D}^5 \times \mathbb{D}^5$, meaning the disk bundle over $\mathbb{D}^5$ looks like a trivial bundle. This always exists and gives rise to a standard two-chart trivialisation of the tangent bundle of $\mathbb{S}^5$, such that it is entirely characterised by the clutching map $S^4 \to \mathrm{SO}(5)$. Let $W := E \cup_{\mathbb{D}^5 \times \mathbb{D}^5} E$ such that the gluing map swaps the sphere- and bundle-coordinate. This is a standard plumbing construction. After smoothing corners, we may assume that W is a smooth 10-manifold with non-empty boundary ∂W.

Theorem 2.1. *The boundary of W is homeomorphic to $\mathbb{S}^9$.*

Proof. This is due to Milnor, see [Mil59], and uses Morse-theory. Given an explicit description of the gluing map, Milnor constructs a Morse function with only two critical points. □

Definition 2.2. The Kervaire manifold M_0 is defined as $M_0 := W \cup_{\partial W} \mathbb{D}^{10}$. Since $\partial W \cong \mathbb{S}^9$, we can attach a cone to W to obtain the closed, 4-connected 10-manifold M_0.

The claim is that M_0 does not admit any smooth structure. For this, we point to Theorem 3.1, which states that for any 4-connected, closed, smooth 10-manifold M, the Kervaire invariant satisfies $\Phi(M) = 0$. Clearly, the manifold M_0 carries a CW-structure which comes from a CW-structure of $T\mathbb{S}^5$. We will now compute the Kervaire invariant of M_0.

Theorem 2.3. *The Kervaire invariant $\Phi(M_0)$ of M_0 is 1.*

Proof. We calculate the cohomology of M_0. Note that we are attaching a 10-cell $\mathbb{D}^{10}$ to W to obtain M_0. Therefore, $H^k(M_0; \mathbb{Z}/2) \cong H^k(W; \mathbb{Z}/2)$ for $0 \leq k \leq 9$. Since M_0 is 4-connected, the only interesting cohomology group is $H^5(M_0; \mathbb{Z}/2)$. The two generators α and β are represented by the two 0-sections of the two copies of $d: E \to \mathbb{S}^5$ by Poincaré duality. Since they intersect transversally in a single point, this forms a symplectic basis. Let $a: \mathbb{S}^5 \to M_0$ be dual to α. This completely lies in the smooth part given by W, hence we can form the normal bundle of the embedded sphere. This is just given by $d: E \to \mathbb{S}^5$. We consider the map $P: M_0 \to \mathrm{Th}(d)$ given by collapsing $M_0 \setminus \dot{E}$ to the basepoint.

Let U again be the space formed from $\mathbb{S}^5$ by killing all homotopy groups of degree at least 9. We claim that $\mathrm{Th}(d) \simeq U^{(10)}$. As $\pi_9(\mathbb{S}^5) \cong \mathbb{Z}/2$, we only need to attach one 10-cell to obtain $U^{(10)}$. Of course, the attaching map is given by the Whitehead-product $[s_5, s_5]: \mathbb{S}^9 \to \mathbb{S}^5$ as this generates $\pi_9(\mathbb{S}^5)$. $\mathrm{Th}(d)$ has only one 5-cell and one 10-cell. This comes from pulling back the characteristic maps $\mathbb{D}^0 \to \mathbb{S}^5$ and $\mathbb{D}^5 \to \mathbb{S}^5$. Since the cells are contractible, the pullback is a product, giving a 5-cell and a 10-cell. As $d: E \to \mathbb{S}^5$ is non-trivial, the attaching map must be homotopic to the Whitehead-product, hence $\mathrm{Th}(d) \simeq U^{(10)}$.

Therefore, we get a map $f: M_0 \to U$ by $P: M_0 \to \mathrm{Th}(d)$, identifying $\mathrm{Th}(d) \simeq U^{(10)}$ and post-composing with the inclusion $i: U^{(10)} \hookrightarrow U$. We get the following commuting diagram.

$$
\begin{tikzcd}
u_5 \arrow[rrr, mapsto] \arrow[dd, mapsto] & & & c(u_5) \arrow[dr, equal] & \\
& H^5(U;\mathbb{Z}) \arrow[r, "c"] \arrow[d, "\cong"'] & H^{10}(U;\mathbb{Z}/2) \arrow[d, "i^*"] & & u_{10} \arrow[d, mapsto] \\
i_5 \arrow[rrrr, mapsto, bend right=10] \arrow[dd, mapsto] & H^5(\mathrm{Th}(d);\mathbb{Z}) \arrow[r, "c"] \arrow[d, "P^*"'] & H^{10}(\mathrm{Th}(d);\mathbb{Z}/2) \arrow[d, "\cong"] & & c(i_5) \arrow[dd, mapsto] \\
& H^5(M_0;\mathbb{Z}) \arrow[r, "c"] & H^{10}(M_0;\mathbb{Z}/2) & & \\
\alpha \arrow[rrrr, mapsto] & & & & c(\alpha)
\end{tikzcd}
$$

Here, u_5 generates $H^5(U;\mathbb{Z})$ and u_{10} generates $H^{10}(U;\mathbb{Z}/2)$. Notice that $f^*(u_5) = \alpha$, and $c(u_5)$ must be a generator by definition of the Kervaire class. Namely, $c(u_5) = [f(\partial e^{10})] = [s, s]$ as the attaching map is given by the Whitehead product. By the cofibration sequence

$$\ldots \longrightarrow 0 \longrightarrow H^{10}(U^{(11)};\mathbb{Z}/2) \xhookrightarrow{i^*} H^{10}(U^{(10)};\mathbb{Z}/2) \longrightarrow H^{11}(\textstyle\bigvee_{i=1}^n \mathbb{S}^{11};\mathbb{Z}/2) \longrightarrow \ldots$$

we can see that i^* sends the generator u_{10} to a generator of $H^{10}(\mathrm{Th}(d);\mathbb{Z}/2)$. By the isomorphism on the right-hand side, $f^*(u_{10}) = c(\alpha)$ must generate $H^{10}(M_0;\mathbb{Z}/2) \cong \mathbb{Z}/2$. This argument can be repeated for the second generator β of $H^5(M_0;\mathbb{Z}/2)$, resulting in $c(\alpha) = c(\beta) = 1$. Therefore, $\Phi(M_0) = c(\alpha)c(\beta) = 1$. □

Combining this result with Theorem 3.1 yields the following corollaries.

Corollary 2.4. *The Kervaire manifold M_0 does not admit any smooth structure.*

Corollary 2.5. *The boundary ∂W of W is homeomorphic to $\mathbb{S}^9$ but not diffeomorphic.*

Proof. If it were, the Kervaire manifold M_0 would be smooth, as $\mathbb{D}^{10}$ is smooth. □

3 Vanishing of the Kervaire Invariant

This section is devoted to the proof of the following theorem.

Theorem 3.1. *Let M be a 4-connected, closed, smooth 10-manifold. Then its Kervaire-invariant $\Phi(M)$ vanishes.*

The proof consists of the following lemmata.

Lemma 3.2. *Any 4-connected, closed, smooth 10-manifold admits a framing.*

Proof. We follow [Ker60]. Let M be a 4-connected, closed, smooth 10-manifold. Let $i : M \hookrightarrow \mathbb{R}^{10+n}$ be an embedding, with n large. Since M is smooth, it admits a CW-structure. We will use obstruction theory to show that the normal bundle $\nu : M \to \mathrm{SO}(n)$ of M in $\mathbb{R}^{10+n}$ is trivial. Since $\pi_4(\mathrm{SO}(n)) = 0$ and M is 4-connected, we get $H^{k+1}(M;\pi_k(\mathrm{SO}(n))) = 0$ for $0 \leq k \leq 8$. Therefore, the only possibly non-trivial obstruction class to constructing a field of normal n-frames f_n is $c(\nu, \mathsf{f}_n) \in H^{10}(M;\pi_9(\mathrm{SO}(n))) \cong \pi_9(\mathrm{SO}(n))$. In [KM58], this has been identified with the kernel of the Hopf-Whitehead homomorphism $J_9 : \pi_9(\mathrm{SO}(n)) \to \pi_9(\mathbb{S})$. This is a monomorphism, hence the obstruction class vanishes. This concludes the proof. □

Lemma 3.3. *The Kervaire invariant defines a homomorphism* $\Omega^{\mathrm{fr}}_{2k} \to \mathbb{Z}/2$.

Proof. In [Bro69], Browder shows that a framed $2k$-dimensional manifold which is cobordant to the empty set has vanishing Kervaire invariant. Let M and N be two framed $2k$-manifolds. Using Mayer-Vietoris, it is easy to see that $H^k(M\#N;\mathbb{Z}/2) \cong H^k(M;\mathbb{Z}/2) \oplus H^k(N;\mathbb{Z}/2)$. Picking a symplectic basis for each summand gives a symplectic basis for $H^k(M\#N;\mathbb{Z}/2)$. Using naturality of the Kervaire form and the collapse maps $M\#N \to M$ and $M\#N \to N$ we get the following commutative diagram.

$$\begin{tikzcd}
H^k(M;\mathbb{Z}/2) \arrow[r, "i_1"] \arrow[d, "c"'] & H^k(M\#N;\mathbb{Z}/2) \arrow[d, "c"'] & H^k(N;\mathbb{Z}/2) \arrow[l, "i_2"'] \arrow[d, "c"] \\
H^{2k}(M;\mathbb{Z}/2) \arrow[r] & H^{2k}(M\#N;\mathbb{Z}/2) & H^{2k}(N;\mathbb{Z}/2) \arrow[l]
\end{tikzcd}$$

We find that $\Phi(M\#N) = \Phi(M) + \Phi(N)$. □

Using the Pontrjagin-Thom isomorphism $\Omega^{\mathrm{fr}}_* \to \pi_*(\mathbb{S})$, we have a homomorphism $\Phi : \pi_{10}(\mathbb{S}) \to \mathbb{Z}/2$ which kills all elements of odd order. It is therefore left to examine the homomorphism on the 2-components.

Lemma 3.4. *Pre-composition with the Hopf map* $\eta \in \pi_1(\mathbb{S})$ *induces a surjection* $\pi_9(\mathbb{S})_{(2)} \twoheadrightarrow \pi_{10}(\mathbb{S})_{(2)}$.

Proof. This can be easily seen by examining the Adams spectral sequence. The only groups that show up in the 10-stem on the E_2-page of the Adams spectral sequence are the following. Let P be the v_1-periodicity element. Simply for degree reasons, Ph_1 and Ph_1^2 are permanent cycles, and no other groups show up. □

Lemma 3.5. *Let* $\beta \circ \eta = \alpha \in \pi_{10}(\mathbb{S})$ *with* $\beta \in \pi_9(\mathbb{S})$. *By the Pontrjagin-Thom isomorphism,* α *is represented by a homotopy* 10*-sphere* Σ^{10}.

Note that any homotopy 10-sphere is homeomorphic to $\mathbb{S}^{10}$. This has been proven by Smale, [Sma61], using the h-cobordism theorem.

Proof. This heavily uses early results in surgery theory. For details, see [Ker60]. Kervaire shows that any framed 9-manifold M representing β is framed cobordant to a homotopy 9-sphere Σ^9 by performing framed surgery. Therefore, α is represented by $\Sigma^9 \times \mathbb{S}^1$. Once again using framed surgery theory, one can easily kill $\pi_1(\Sigma^9 \times \mathbb{S}^1)$ and obtains a framed cobordism to a homotopy 10-sphere Σ^{10}. □

Proof of Theorem 3.1. Combining the previous lemmata, since $H^k(\Sigma^{10};\mathbb{Z}/2) = 0$, the Kervaire invariant vanishes. □

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