The Rokhlin Invariant and Homology Cobordism

Maximilian Hans

4th October 2022

Abstract

The well-known basic relation between the Rokhlin invariant and homology cobordism will be established. We will consider the group $\Theta_{\mathbb{Z}}^3$ of integer homoloy spheres up to homology cobordism. Surprisingly, this group plays a key role in disproving the triangulation conjecture. We mostly follow [Sav02].

1 Homology Cobordism

Definition 1.1 (H-cobordism). Let Σ_1 and Σ_2 be two integer homology spheres. We call Σ_1 and Σ_2 homology cobordant, or *H*-cobordant, if there exists a smooth compact oriented manifold *W* of dimension 4 such that $\partial W = \Sigma_1 \amalg -\Sigma_2$ and the inclusions $i_1 : \Sigma_1 \hookrightarrow W$, $i_2 : \Sigma_2 \hookrightarrow W$ induce an isomorphism in homology. We can choose *W* to carry a spin structure.

Remark 1.2. This defines an equivalence relation on the set of all integer homology spheres.

Definition 1.3 (H-cobordism group). We denote $\Theta_{\mathbb{Z}}^3$ to be the set of all such equivalence classes. Under the operation of connected sum, this defines an abelian group. For an integer homology sphere Σ , we write $[\![\Sigma]\!]_H$ as its equivalence class in $\Theta_{\mathbb{Z}}^3$.

We need to check whether that is truly the case. If we show that the connected sum gives $\Theta_{\mathbb{Z}}^3$ a group structure, this group being abelian follows immediately. For that, we need some results.

Lemma 1.4. Let Σ be an integer homology sphere. Σ is H-cobordant to \mathbb{S}^3 if and only if there exists a smooth 4-manifold W with boundary Σ and $H_{\bullet}(W;\mathbb{Z}) \cong H_{\bullet}(\mathbb{D}^4;\mathbb{Z})$.

Proof. Assume there exists such a 4-manifold W. We can smoothly embed a \mathbb{D}^4 in the interior of $W, i : \mathbb{D}^4 \hookrightarrow W$. Then $W \setminus i(\mathbb{D}^4)$ is an H-cobordism of Σ and \mathbb{S}^3 , as removing \mathbb{D}^4 leaves one more boundary component, namely \mathbb{S}^3 . Conversely, assume Σ and \mathbb{S}^3 are H-cobordant. Attaching \mathbb{D}^4 to the boundary component \mathbb{S}^3 yields a 4-manifold W with $\partial W = \Sigma$ and $H_{\bullet}(W; \mathbb{Z}) \cong H_{\bullet}(\mathbb{D}^4; \mathbb{Z})$. Figure 1 shows a sketch of the construction.



Figure 1: Schematic picture of the construction above.

Proposition 1.5. Let Σ_1, Σ_2 be integer homology spheres. Then $\Sigma_1 \# \Sigma_2$ is an integer homology sphere as well.

Proof. First of all, $\Sigma_1 \# \Sigma_2$ is again a closed connected and oriented 3-manifold. We consider the long exact sequence of the pair $(\Sigma_1 \# \Sigma_2, \mathbb{S}^2)$, with \mathbb{S}^2 obtained from the construction of the connected sum. Note, that we have the following relation between homology of good pairs and reduced homology.

$$H_n(\Sigma_1 \# \Sigma_2, \mathbb{S}^2; \mathbb{Z}) \cong \tilde{H}_n((\Sigma_1 \# \Sigma_2) / \mathbb{S}^2; \mathbb{Z})$$
$$\cong \tilde{H}_n(\Sigma_1 \vee \Sigma_2; \mathbb{Z}) \cong \tilde{H}_n(\Sigma_1; \mathbb{Z}) \oplus \tilde{H}_n(\Sigma_2; \mathbb{Z})$$

The long exact sequence of pairs in homology now yields the following.

$$0 \longrightarrow 0 \longrightarrow H_3(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow H_2(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_1(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow H_0(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \longrightarrow 0 \longrightarrow 0$$

We can directly see that $H_0(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \cong \mathbb{Z}$ and $H_1(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \cong 0$. Using Poincaré Duality and the Universal Coefficient Theorem for cohomology yields $H_2(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \cong 0$. As $\Sigma_1 \# \Sigma_2$ is oriented, we already know that $H_3(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \cong \mathbb{Z}$. One can also see it from the long exact sequence above, as the short exact sequence splits.

$$0 \to H_3(\Sigma_1 \# \Sigma_2; \mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

Proposition 1.6. $\Theta^3_{\mathbb{Z}}$ is an abelian group under the operation of connected sum.

Proof. The neutral element is given by the sphere \mathbb{S}^3 . This follows from the fact that $\mathbb{S}^3 \# \mathbb{S}^3 \cong \mathbb{S}^3$, hence $[\mathbb{S}^3]_H = 0$. Inverses are given by reversing orientation. We need to show that for Σ , an integer homology sphere, $\Sigma \# - \Sigma$ is homology cobordant to \mathbb{S}^3 . Let $W = \Sigma \times [0,1]$. This defines an Hcobordism $W : \Sigma \to \Sigma$. Let $\gamma : [0,1] \to W$ be a path connecting $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$. Consider the thickening of γ , $U(\gamma) \cong \mathbb{D}^3 \times \gamma \subseteq W$. Note, that $\partial U(\gamma) \cong \mathbb{S}^2 \times \gamma$. This is closed, and $W \setminus \dot{U}(\gamma)$ is closed as well. Here, $\dot{U}(\gamma)$ denotes $U(\gamma) \setminus \partial U(\gamma)$. The boundary of $W \setminus \dot{U}(\gamma)$ now consists of the connected sum of three components, namely $\Sigma \# (\mathbb{S}^2 \times \gamma) \# - \Sigma$. A quick calculation yields the following result.

$$H_n(W, W \setminus \dot{U}(\gamma); \mathbb{Z}) \cong H_n(W/(W \setminus \dot{U}(\gamma)), *; \mathbb{Z})$$
$$\cong \tilde{H}_n(W/(W \setminus \dot{U}(\gamma)); \mathbb{Z}) \cong \tilde{H}_n(\mathbb{S}^3 \times [0, 1]; \mathbb{Z})$$

The long exact sequence of pairs in homology now yields

$$\cdots \to H_n(W \setminus \dot{U}(\gamma); \mathbb{Z}) \xrightarrow{i_*} H_n(W; \mathbb{Z}) \to \tilde{H}_n(\mathbb{S}^3 \times [0, 1]; \mathbb{Z}) \to \dots$$

and as W is homotopy equivalent to Σ which has the same homology groups as \mathbb{S}^3 , $\mathbb{S}^3 \times [0, 1]$ is homotopy equivalent to \mathbb{S}^3 , we obtain the following exact sequence.

$$0 \longrightarrow H_{3}(W \setminus \dot{U}(\gamma); \mathbb{Z}) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$\xrightarrow{\hookrightarrow} H_{2}(W \setminus \dot{U}(\gamma); \mathbb{Z}) \longrightarrow 0 \longrightarrow 0$$

$$\xrightarrow{\hookrightarrow} H_{1}(W \setminus \dot{U}(\gamma); \mathbb{Z}) \longrightarrow 0 \longrightarrow 0$$

$$\xrightarrow{\hookrightarrow} H_{0}(W \setminus \dot{U}(\gamma); \mathbb{Z})) \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

Going through the sequence, we can deduce $H_n(W \setminus \dot{U}(\gamma); \mathbb{Z}) \cong H_n(\mathbb{D}^4; \mathbb{Z})$. Hence by Lemma 1.4, $\Sigma \# - \Sigma$ is *H*-cobordant to \mathbb{S}^3 , and $-\llbracket \Sigma \rrbracket_H = \llbracket -\Sigma \rrbracket_H$.

Proposition 1.7. The map $\Theta_{\mathbb{Z}}^3 \to \mathbb{Z}/2\mathbb{Z}$ given by the Rokhlin invariant is a surjective group homomorphism.

Proof. Let Σ_1 and Σ_2 be two integer homology spheres. Consider $\Sigma_1 \# \Sigma_2$. By Proposition 1.5, this is again an integer homology sphere. Let W_1 be a compact connected smooth spin 4-manifold with boundary Σ_1 , W_2 one which Σ_2 bounds. Then $W_1 \natural W_2$ has boundary $\Sigma_1 \# \Sigma_2$ and its intersection form is given by $Q_{W_1} \oplus Q_{W_2}$. The signature is additive. Therefore, $\mu(\Sigma_1 \# \Sigma_2) = \mu(\Sigma_1) + \mu(\Sigma_2) \mod 2$. We already know that $\mu(\mathbb{S}^3) = 0$. Suppose Σ is an integer homology sphere that is *H*-cobordant to \mathbb{S}^3 via a compact smooth 4-manifold W with $H_{\bullet}(W; \mathbb{Z}) \cong H_{\bullet}(\mathbb{D}^4; \mathbb{Z})$. Its first homology group is trivial and the intersection form Q_W is empty, hence has even signature, and a spin structure. Attaching \mathbb{D}^4 to the boundary component \mathbb{S}^3 yields the following.

$$\mu(\Sigma) = \frac{1}{8}\sigma(W) = 0 \mod 2$$

Therefore, the Rokhlin invariant truly defines a group homomorphism $\Theta_{\mathbb{Z}}^3 \to \mathbb{Z}/2\mathbb{Z}$. As $\mu(\Sigma(2,3,5)) = 1$, it is a surjective mapping.

We will now give an argument on why the Poincaré homology sphere $\Sigma(2,3,5)$ has infinite order in $\Theta^3_{\mathbb{Z}}$. For that, we need the following classical result by Donaldson.

Theorem 1.8. (Donaldson) Let W be a closed smooth oriented 4-manifold. If the intersection form Q_W is positive or negative definite, then it can be diagonalised over Z to the identity matrix or its additive inverse, respectively.

Proof. See [Don87] for a proof.

Proposition 1.9. $\Sigma(2,3,5)$ has infinite order in $\Theta^3_{\mathbb{Z}}$.

Proof. Recall that $\Sigma(2,3,5)$ bounds a smooth closed oriented 4-manifold W with intersection form E_8 . Consider $\#_{i=1}^n \Sigma(2,3,5)$ for $n \ge 1$. This bounds the smooth closed oriented manifold 4-manifold $\natural_{i=1}^n W$. Its intersection form is isomorphic to $\bigoplus_{i=1}^n E_8$. Let us now assume that there exists a certain $k \in \mathbb{N}$ such that $[\![\#_{i=1}^k \Sigma(2,3,5)]\!]_H$ is trivial, meaning homology cobordant to \mathbb{S}^3 . Let N denote this cobordism and consider the manifold $\natural_{i=1}^k W \cup_{\Sigma} -N$. This is a smooth closed oriented manifold of dimension 4 with intersection form $\bigoplus_{i=1}^k E_8$ which is definite but not diagonalisable over \mathbb{Z} . This contradicts Donaldson's Theorem, proving the claim.

This also implies that $\mathbb{Z} \leq \Theta_{\mathbb{Z}}^3$. We now will state some interesting known properties of the homology cobordism group. This will be held informally, proofs will be omitted.

Theorem 1.10 (Furuta, Fintushel and Stern). $\mathbb{Z}^{\infty} \leq \Theta_{\mathbb{Z}}^3$.

Proof. The classes $[\![\Sigma(2,3,6n-1)]\!]_H$ for $n \ge 1$ given by Brieskorn homology spheres are linearly independent. For example, see [Fur90].

2 A brief word on the Triangulation conjecture

In 2016, Manolescu published a paper in which he disproved the triangulation conjecture.

Conjecture 2.1 (Triangulation conjecture). Every topological manifold is triangulable.

Up to dimension 3, this is indeed true. Casson and Freedman disproved the triangulation conjecture in dimension 4, the E_8 -manifold for example is not triangulable. The following result is due to the work of Matumoto, Galewski, and Stern.

Theorem 2.2 (Matumoto, Galewski and Stern). For dimension greater or equal than 5, every topological manifold M with empty boundary is triangulable if and only if the following short exact sequence splits.

$$0 \to ker(\mu) \to \Theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2\mathbb{Z} \to 0$$

The map μ is given by the Rokhlin invariant.

Proof. See [GS80] and [Mat78].

This means that any such manifold is triangulable if and only if there exists an integer homology sphere Σ such that $\mu(\Sigma) = 1$ and $\Sigma \# \Sigma$ is homology cobordant to \mathbb{S}^3 .

Theorem 2.3 (Manolescu). There is no 2-torsion in $\Theta_{\mathbb{Z}}^3$.

Proof. See [Man16].

Hence, the short exact sequence in Theorem 2.2 does not split, and there do exist manifolds of dimension greater or equal than 5 which are not triangulable, disproving the triangulation conjecture.

Regarding the group $\Theta_{\mathbb{Z}}^3$, an interesting question to think about would be if it is torsion-free. To this date, this is an open question.

References

- [Mat78] T. Matumoto. "Triangulation of Manifolds". In: Proceedings of Symposia in Pure Mathematics 32 (1978), pp. 3–6.
- [GS80] David E. Galewski and Ronald J. Stern. "Classification of Simplicial Triangulations of Topological Manifolds". In: Annals of Mathematics 111.1 (1980), pp. 1–34.
- [Don87] S. K. Donaldson. "The orientation of Yang-Mills moduli spaces and 4-manifold topology". In: Journal of Differential Geometry 26.3 (1987), pp. 397–428.
- [Fur90] M. Furuta. "Homology cobordism group of homology 3-spheres". In: Inventiones mathematicae 100 (1990), pp. 339–355.
- [Sav02] Nikolai Saveliev. Invariants for homology 3-spheres. 2002.
- [Man16] Ciprian Manolescu. "Pin(2)-equivariant Seiberg-Witten Floer Homology and the Triangulation Conjecture". In: Journal of the American Mathematical Society 29.1 (2016), pp. 147– 176.